7/25/68

WEAK EXTENSIONS OF OPERATORS WITH EMPHASIS ON INTEGRALS

A THESIS

Presented to

The Faculty of the Division of Graduate

Studies and Research

by

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In Partial Fulfillment

of the Requirements for the Degree

Doctor of Philosophy

in the School of Mathematics

Georgia Institute of Technology

June, 1970

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Approved:

Chairman

Date approved by Chairman: July 28, 1970

ACKNOWLEDGMENTS

I wish to thank my thesis advisor, Dr. S. H. Coleman, for his reading and criticizing the many preliminary drafts of this thesis. In addition, I would like to thank Dr. J. W. Walker and Dr. S. H. Coleman for their criticism of part of Chapter III and their advice which lead to using category theory as the foundation of the development there. I would also like to thank Dr. E. R. Immel for his reading of the thesis.

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CHAPTER I

INTRODUCTION

The objectives of this research are: (1) to define a method of extending an operator I, defined for functions mapping a space X into a space Z, to an operator $\hat{I}$, defined for functions mapping X into a (new) space Y, and to study the properties of $\hat{I}$; and (2) to apply this technique of extension to the case where I is the Lebesgue integral, Z is the set of real numbers or the set of complex numbers, and Y is an arbitrary non-empty set, or a topological space, or a partially ordered set, or a group topological space, or a partially ordered group topological space, or a topological group, or a Banach space. We require that the values of $\hat{I}$ be in Y or in some extension of Y.

The main purpose of this research is the generalization of integration and it is in the spirit of McShane's generalization [1] of integration from an order point of view.

General properties that we would like the extension $\hat{I}$ to possess are:

(P1) $\hat{I}$ inherits most of the basic properties of I;

(P2) if $Y=Z$, then $\hat{I}=I$ on their identical domains;

(P3) if ϕ maps X into Y and T maps Y into Q, then $\hat{I}\big(T(\phi)\big) = T\big(\hat{I}(\phi)\big)$;

(P4) every operator I', defined for functions on X into Y, is an $\hat{I}$ for some I; and

(P5) if I is the Lebesgue integral and Y is a Banach space, then $\hat{I}$ contains some of the existing generalizations of the Lebesgue integral to vector-valued functions.

In Chapter III we define the extension $\hat{I}$ of a general operator I and discuss properties (P1) through (P5). Chapters IV and V deal with properties (P1) and (P2) when I is the Lebesgue integral.

In order to obtain a method of extending a general operator I, we first studied the techniques already used in generalizing the Lebesgue integral. Most of the generalizations of the Lebesgue integral require a structure on the range space (Y) of the functions to be integrated which is more than just a topology. They usually hypothesize that either addition and scalar multiplication are defined on Y, or that an ordering exists on Y. We were looking for a method that could be generalized to the case that Y is a topological space or just an arbitrary non-empty set. This means that the technique must have the property that it can be extended so that the definition of $\hat{I}$ is independent of any structure on Y. Then we can put various structures on Y and deduce the resulting appropriate properties of $\hat{I}$.

The most important generalizations of the Lebesgue integral to functions mapping a measure space into a linear topological space Y are, in their order of generality, the Bochner integral [4], the Birkhoff integral [5], the Pettis integral [2], the Dunford integral [3], the Phillips integral [6], and the Rickart integral [7].* The first four

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*See Hildebrandt [8] for a discussion of these integrals and many more generalizations of the Lebesgue integral.

integrals require that Y be a Banach space over the reals, and Y is specified as a locally convex linear topological space in the case of the last two integrals.

The first question, then, is which of these integrals will generalize to the case in which Y is a topological group? That is, if we take the scalars away, will more or less the same procedure go through? In the Bochner, Birkhoff, and Phillips integrals scalar multiplication is essential because each needs to consider sums like $\sum \phi(x_i)\mu(E_i)$ where $\mu(E_i)$ is the measure of a set (a scalar) and $\phi(x_i)$ is a vector, or a set of vectors, in Y. Rickart's integral has been generalized to the case that Y is a topological group ([9],[10]). The basic idea here was to do away with the measure function entirely, and therefore not need scalar multiplication.

Rickart's method, and its generalization to Y a topological group, will not extend to the case that Y is a topological space because addition of elements in Y is essential in this technique.

The procedure used by Pettis and Dunford can be generalized to the case that Y is a topological group or a topological space. Both used members of the dual of Y (all the continuous linear maps of Y into the reals; the members of the dual are called <u>functionals</u>) to transfer the range space of the functions to be integrated to the reals. That is, if ϕ maps the measure space X into Y, they then would consider the set $\{f(\phi) | f \text{ in the dual of } Y\}$. If Y is a topological group we do not have a range space for the functionals automatically supplied. For the moment, let Z, possibly the reals or the complex numbers, be the range space of the functionals. Therefore we propose to use functionals, in exactly the same way Pettis and Dunford did, to extend the Lebesgue integral to functions mapping a measure space into Y, except that we will vary our types of functionals to match the properties of Y. For example, if Y is a topological group (no scalars) we can only require the functionals to be continuous homomorphisms on Y into Z; if Y is a topological space (no addition) the functionals could be continuous maps of Y into Z; and if Y is an arbitrary non-empty set the functionals are functions on Y into Z. We will also vary the space Z to match the properties of Y. This method can be used to extend a general operator, and if Y is a Banach space and I is the Lebesgue integral we will see that $\hat{I}$ contains the integrals of Bochner, Birkhoff, Pettis, and Dunford.

In Chapter II we define the Pettis and Dunford integrals, state a few of their basic properties, and look at the relationship between these two integrals.

In Chapter III we generalize the ideas of Pettis and Dunford and construct the extension of a general operator and develop its properties. Here the spaces Y and Z are usually arbitrary non-empty sets and the functionals are just functions mapping Y into Z. We also discuss how the extension $\hat{I}$ is related to the Pettis and Dunford integrals when Y is a Banach space and I is the Lebesgue integral. For the rest of the paper, we specialize to the case in which I is the Lebesgue integral.

In Chapter IV we use the reals under addition as the range space of the functionals. Here we start with Y an arbitrary non-empty set, and step-by-step add structure to Y and derive the appropriate properties of $\hat{I}$. Whenever possible the functionals will be continuous order preserving homomorphisms of Y into the reals. We show that when Y is a partially ordered group topological space,* $\hat{I}$ inherits most of the basic properties of the Lebesgue integral.

In Chapter V we use the complex numbers under multiplication for the range space of the functionals. This is a natural choice for the range space because of Pontrjagin's duality theorem (Theorem 1 of Chapter V). Here we show that, when the complex numbers under multiplication is used for the range space of the functionals, there is a useful** generalization of the Lebesgue integral if and only if the measure function takes on only two values, zero and one.

Chapter VI consists of three examples. Each example deals with the extension of the Lebesgue integral when the reals under addition is used as the range space of the functionals. In Chapter IV we always assume that the domain of the extension is non-empty and that there are many non-trivial functionals. These examples show that this is the case for three different spaces Y. The last example in Chapter VI deals with an integral for set-valued functions. There has been considerable

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*There is no relationship assumed between the order relation, addition, and the topology.

**The sense in which the generalization is useful is indicated by Theorems 2, 3, 4, and 5 of Chapter V.

interest recently in extending results to set-valued functions, and an integral has been studied for these functions ([11],[12],[13],[14]). This integral has been called Aumann's integral. Example 3 shows that our extension of the Lebesgue integral contains Aumann's integral when Y consists of all non-empty subsets of real numbers.

There are three appendices. Appendix A gives the proof of a theorem that is needed in Chapter IV. In Appendix B we collect together certain facts about the Lebesgue integral of a complex-valued function with respect to a complex measure that are needed in Chapter V. In Appendix C we prove four theorems that are used in Example 3 of Chapter VI.

CHAPTER II

THE PETTIS AND DUNFORD INTEGRALS

In this chapter we will define the Pettis [2] and Dunford [3] integrals, state a few of their properties, and look at the relationship between these two integrals. In the next chapter we generalize the ideas of Pettis and Dunford and construct the extension of a general operator.

The Pettis Integral

Let $(X,\mathcal{A},\mu)$ be a measure space, $\mathcal{A}$ a σ-algebra of subsets of X, and μ a finite measure on $\mathcal{A}$. Y will be a Banach space over the reals, $Y'(Y'')$ denotes all the continuous, linear, real-valued functions on $Y(Y')$, and $L(\psi,E)$ stands for the Lebesgue integral of real-valued ψ over $E\epsilon\mathcal{A}$ with respect to μ.

Definition 1

(a) A function ϕ on X into Y is weakly measurable if and only if $f(\phi)$ is measurable for every $f\epsilon Y'$.

(b) A function ϕ on X into Y is Pettis integrable over $E\epsilon\mathcal{A}$ if and only if ϕ is weakly measurable and there exists a $y_E\epsilon Y$ such that $f(y_E) = L\big(f(\phi),E\big)$ for every $f\epsilon Y'$. If so, the Pettis integral of ϕ over E with respect to μ, written $P(\phi,E)$, is defined to be y_E.

(c) A function ϕ on X into Y is <u>Pettis integrable</u> if and only if it is Pettis integrable over E for every $E\epsilon A$.*

Some of the immediate consequences of Definition 1 are:

(a) the integral is a linear function of the integrand;

(b) if ϕ_1 is Pettis integrable, $\phi_1 = \phi_2$ a.e., and ϕ_2 is weakly measurable,** then ϕ_2 is Pettis integrable and $P(\phi_1,E) = P(\phi_2,E)$ for every $E\epsilon A$;

(c) if Y is the set of real numbers the Pettis integral coincides with the Lebesgue integral; and

(d) if $\{A_1,\cdots,A_n\}$ is a measurable partition of X and $\phi(x) = y_i\epsilon Y$ for $x\epsilon A_i$, $i=1,2,\cdots,n$, then ϕ is Pettis integrable and $P(\phi,E) = \sum_{i=1}^{n} y_i\mu(E\cap A_i)$ for every $E\epsilon A$.

We wish to look at the Pettis integral when Y is a Hilbert space, but first we need the following theorem, a proof of which is given in Hille and Phillips ([15], p.77).

<u>Theorem 1</u>. Let (X,A,μ) be a σ-finite measure space, ϕ a function on X into Y which is weakly measurable and $|L(f(\phi),X)| < +\infty$ for every $f\epsilon Y'$. Then the equation $F(f) = L(f(\phi),X)$, for every $f\epsilon Y'$, defines a function F on Y' into the reals and $F\epsilon Y''$.

<u>Y a Hilbert Space</u>

Let Y be a Hilbert space over the reals. Then, by the Riesz

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*Pettis ([2], p.302) has observed that it may happen that ϕ is Pettis integrable over X but ϕ is not Pettis integrable.

**This can be omitted if μ is a complete measure.

representation theorem, if $f \in Y'$ there exists a $y_o \in Y$ such that $f(y) = (y, y_o)$ for every $y \in Y$. Therefore every $f \in Y'$ is of the form $(\cdot, y)$ for some $y \in Y$.

Now assume that ϕ is a function on X into Y so that $(\phi(x), y)$ is measurable in x for every $y \in Y$, and $|L((\phi(x), y), X)| < +\infty$ for every $y \in Y$. By Theorem 1, the equation $f(y) = L((\phi(x), y), E)$, for every $y \in Y$ and each $E \in A$, defines a $f \in Y'$. By the Riesz representation theorem there is a $y_E \in Y$ such that $f(y) = (y, y_E)$ for every $y \in Y$. Hence $(y, y_E) = L((\phi(x), y), E)$ for every $y \in Y$ and each $E \in A$. From Definition 1 we have $P(\phi, E) = y_E$, so $(y, P(\phi, E)) = L((\phi(x), y), E)$ for every $y \in Y$ and each $E \in A$.

Relation to Other Integrals

Let Q be another Banach space over the reals, and T a continuous linear map of Y into Q. Pettis has shown ([2], p.280) that if ϕ is Pettis integrable, then so is $T(\phi)$, and*

$$P(T(\phi), E) = T(P(\phi, E)) \quad \text{for each } E \in A. \tag{1}$$

Pettis uses this property of his integral to show it contains the Bochner and Birkhoff integrals ([2], p.281). In fact, if S is any integral defined for functions mapping X into Y that reduces to the Lebesgue integral when Y is the real numbers and if the S integral has the property of Equation (1), then every S integrable function is Pettis integrable and the integral values agree.**

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*We generalize this for a general operator I in Theorem 4 in Chapter III.

**This is generalized by Theorem 6 in Chapter III.

Y a Reflexive Banach Space

A useful characterization of Pettis integrable functions is not known ([15], pp.77-78; [16]), but if Y is reflexive ($Y=Y''$), then ϕ is Pettis integrable if and only if ϕ is weakly measurable and $|L(f(\phi),X)| < +\infty$ for every f in Y'. This follows from the fact (Theorem 1) that the formula

$$F(f) = L(f(\phi),E) \quad \text{for every } f\epsilon Y' \tag{2}$$

defines a $F\epsilon Y'' = Y$ for each $E\epsilon A$ whenever $|L(f(\phi),X)| < +\infty$ for all $f\epsilon Y'$ and Y is reflexive. This also leads us to Dunford's integral because Dunford's integral of ϕ will be equal to that F in Y'' defined by Equation (2).

The Dunford Integral

Now we assume that A is a σ-algebra of subsets of X and μ is a finite signed measure. We will write μ^+ and μ^- for the positive and negative parts of μ(μ^+ and $-\mu^-$ are finite measures) and $|\mu| = \mu^+ - \mu^-$. If $L(\psi d\mu^+,E)$ and $L(\psi d(-\mu^-),E)$ are the Lebesgue integrals of ψ over $E\epsilon A$ with respect to measures μ^+ and $-\mu^-$, then the Lebesgue integral of ψ over $E\epsilon A$ with respect to μ, written $L(\psi d\mu,E)$, is defined as $L(\psi d\mu^+,E)$ - $L(\psi d(-\mu^-),E)$ as long as the subtraction is defined. $L(\psi d(|\mu|),E)$ is defined as $L(\psi d\mu^+,E) + L(\psi d(-\mu^-),E)$.

Y is a Banach space over the reals, $Y'(Y'')$ denotes all the continuous, linear, real-valued functions on $Y(Y')$, Γ is a closed linear manifold in Y', and Γ' stands for all continuous, linear, real-valued functions on Γ. R is the set of real numbers.

Definition 2

(a) If $E\epsilon A$ and $1\le p < +\infty$, then $L_p(E) = \{\psi\colon E\to R \mid \psi$ is measurable and $L(|\psi|^p d(|\mu|),E) < +\infty\}$.

(b) If $E\epsilon A$, then $L_\infty(E) = \{\psi\colon E\to R \mid \psi$ is measurable and there exists an $A\subset E$, $A\epsilon A$, $|\mu|(A) = 0$, and $|\psi| \le M$, some $M > 0$, on $E - A\}$.

(c) If $E\epsilon A$ and $1\le p \le +\infty$, then $L_p(E)[Y,\Gamma] = \{\phi\colon E\to Y \mid f(\phi)\epsilon L_p(E)$ for every $f\epsilon\Gamma\}$.

(d) If $E\epsilon A$, $1\le p \le +\infty$, and $\phi\epsilon L_p(E)[Y,\Gamma]$, then the Dunford integral of ϕ over E with respect to μ, written $Du(\phi,E)$, is defined to be that $F\colon \Gamma\to R$ where $F(f) = L(f(\phi)d\mu,E)$ for every $f\epsilon\Gamma$.

Some of the basic properties of the Dunford integral are:

(a) if $E\epsilon A$, $1\le p \le +\infty$, and $\phi\epsilon L_p(E)[Y,\Gamma]$, then $Du(\phi,E) \epsilon \Gamma'$;

and

(b) the space $L_p(E)[Y,\Gamma]$ is a normed linear space* and $Du(\cdot,E)$ is a continuous linear operator on $L_p(E)[Y,\Gamma]$ into Γ'.

Relationship with the Pettis Integral

Let $\Gamma=Y'$ and let μ be a finite measure on a σ-algebra A of subsets of X. Then the equation $F(f) = L(f(\phi),X)$ for every $f\epsilon Y'$, $\phi\epsilon L_1(X)[Y,Y']$, defines a F in Y'' (Theorem 1). We have $Du(\phi,X) = F$.

If we define $\lambda(y) = F_y$ for each y in Y, where $F_y(f) = f(y)$ for every f in Y', then $\lambda\colon Y\to Y''$. In general $Du(\phi,X) = F$ cannot be replaced

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*See Dunford ([3], p.339) for the norm on $L_p(E)[Y,\Gamma]$.

by an element in Y $\left(F\notin\lambda(Y)\right)$. When such a replacement can be made we get the Pettis integral and the two integrals agree. If Y is reflexive $\left(Y''=\lambda(Y)\right)$, then the two integrals agree on $L_1(X)[Y,Y']$.

In case Y is not reflexive we have ([2], p.294; [3], pp. 339-340):

(a) if $\phi\epsilon L_p(X)[Y,Y']$, $p>1$, and ϕ is the a.e. limit of simple functions, then ϕ is Pettis integrable and $P(\phi,E) = Du(\phi,E)$ for each $E\epsilon\mathcal{A}$;

(b) if $\phi\epsilon L_1(X)[Y,Y']$, ϕ is the a.e. limit of simple functions, and Y is weakly complete, then ϕ is Pettis integrable and $P(\phi,E) = Du(\phi,E)$ for each $E\epsilon\mathcal{A}$; and

(c) if Y is separable and $p>1$, then $P(\phi,E) = Du(\phi,E)$ for each $E\epsilon\mathcal{A}$ whenever $\phi\epsilon L_p(X)[Y,Y']$.

CHAPTER III

GENERAL PROPERTIES OF THE EXTENSION

In this chapter we will generalize the methods of Pettis and Dunford ([2],[3]) to extend an operator I, defined for functions mapping a space X into a space Z, to an operator $\hat{I}$, defined for functions on X into a (new) space Y.

The first section is concerned with constructing the extension and the question of how it is related to the Pettis and Dunford integrals. In the second section we discuss the following basic properties of the extension: (1) continuity, (2) $\hat{I}$ reduces to I whenever Y=Z, (3) the composition theorem $T\left(\hat{I}(\phi)\right) = \hat{I}\left(T(\phi)\right)$ and how this characterizes $\hat{I}$, and (4) that every operator I', defined for functions on X into Y, whose values are in Y or some extension of Y, is an $\hat{I}$ for some operator I. We introduce category theory in order to discuss properties (2), (3), and (4).

The spaces X, Y, and Z will usually be arbitrary non-empty sets or topological spaces, and I will be a general operator. In this general setting it is difficult to deduce very much about the extension $\hat{I}$, even to the extent of its existence. One can choose X, Y, Z, I, and functionals f (Definition 1) so that the domain of $\hat{I}$ is empty. Therefore, we will always assume that we are working with spaces X, Y, and Z, an operator I, and functionals f, such that the domain of $\hat{I}$ is not empty.

In the next chapter, I will be the Lebesgue integral and Z will be the set of real numbers under addition. There we will see that $\hat{I}$ inherits most of the basic properties of I. In Chapter V, I will again be the Lebesgue integral, but Z will be the set of complex numbers under multiplication.

If we do not specify a particular topology for a function space, then we assume that it has the product topology.

Construction of the Extension

In this section X, Y, and Z are arbitrary non-empty sets. If I is an operator, whose values are in Z, which is defined for certain functions on X into Z, we will extend it to an operator $\hat{I}$, whose values will be in Y or some extension of Y, which is defined for certain functions on X into Y. We will then discuss how this generalizes the methods used by Pettis and Dunford to extend the Lebesgue integral to vector-valued functions.

Definition 1

(a) If J is any function, D(J) is the domain of J and R(J) its range.

(b) If A and B are non-empty sets, then (A,B) is the set of all functions on A into B. An ordered pair of sets will be denoted by $\langle A,B \rangle$.

(c) Y' is a non-empty subset of (Y,Z) called the *dual* of Y. Its elements are written f and they are called *functionals*.

(d) $D(\hat{I}) = \{\phi\epsilon(X,Y)|f(\phi)\epsilon D(I)$ for every $f\epsilon Y'\}$.

(e) (Canonical map) Define $\lambda:Y\to(Y',Z)$ as $\lambda(y) = F_y$, where $F_y(f) = f(y)$ for every $f\epsilon Y'$.

The reason why we call Y' the dual of Y is that the rôle Y' plays in extending I is the same as that played by the dual of a Banach space in Dunford's integral. Since the extension $\hat{I}$ is based on Dunford's integral, the terminology throughout will be borrowed from functional analysis.

We will always write F for members of (Y',Z), ϕ for elements of (X,Y), ψ for functions on X into Z, and F_y for elements of $R(\lambda)$. $D(\hat{I})$ will be the domain of $\hat{I}$, and we always assume that $D(\hat{I})$ is non-empty, since otherwise there is no extension.

If $\phi\epsilon D(\hat{I})$, then the equation $I\big(f(\phi)\big) = F(f)$ for every $f\epsilon Y'$ uniquely defines an element $F\epsilon(Y',Z)$, for each fixed ϕ. Now F depends on ϕ, and we might show this by writing F_ϕ, but we will usually omit the subscript on F. We might define a function $\hat{I}$ on $D(\hat{I})$ into (Y',Z) as $\hat{I}(\phi) = F$ where $I\big(f(\phi)\big) = F(f)$ for every $f\epsilon Y'$. But this F might be in $R(\lambda)$ and then we could identify it with a subset of Y by using the inverse of the canonical map λ. This brings us to the following definitions of $\hat{I}$.

Definition 2

(1) (a) If $\phi\epsilon D(\hat{I})$, $I\big(f(\phi)\big) = F(f)$ for every $f\epsilon Y'$, and $F = F_y\epsilon R(\lambda)$ for some $y\epsilon Y$, then $\hat{I}(\phi) = \lambda^{-1}(F_y)$.

(b) If $\phi\epsilon D(\hat{I})$, $I[f(\phi)] = F(f)$ for every $f\epsilon Y'$, and $F\notin R(\lambda)$, then $\hat{I}(\phi) = F$.

(2) $D_o(\hat{I}) = \{\phi\epsilon D(\hat{I}) | F_\phi \epsilon R(\lambda)$, where $F_\phi(f) = I[f(\phi)]$ for every $f\epsilon Y'\}$.

$D_o(\hat{I})$ is just the set of all those $\phi\epsilon D(\hat{I})$ such that there exists a $y\epsilon Y$ satisfying $I[f(\phi)] = f(y)$ for every $f\epsilon Y'$, since $F_y(f) = f(y)$ for every $f\epsilon Y'$. Also, if $\phi\epsilon D_o(\hat{I})$, then $\hat{I}(\phi) = \lambda^{-1}(F_y) = \{y\epsilon Y | I[f(\phi)] = f(y)$ for every $f\epsilon Y'\}$.

For each $y\epsilon Y$, we will identify F_y and $\lambda^{-1}(F_y)$, and so we will write $\hat{I}(\phi) = F_y$, or $\hat{I}(\phi) = \lambda^{-1}(F_y)$, whichever is more convenient at the time, for each $\phi\epsilon D_o(\hat{I})$. Therefore, $\hat{I}$ maps $D(\hat{I})$, a subset of (X,Y), into (Y',Z), or $\hat{I}$ maps some $\phi\epsilon(X,Y)$ into (Y',Z) and some $\phi\epsilon(X,Y)$ into subsets of Y.

Of course, $D_o(\hat{I})$ may be empty, but at the other extreme, it may happen that $D_o(\hat{I}) = D(\hat{I}) \neq \emptyset$ (see the examples in Chapter VI). When $D_o(\hat{I})$ is non-empty, we are naturally interested in the case in which $\hat{I}$ is point-valued in Y on $D_o(\hat{I})$. This brings us to the next definition.

<u>Definition 3.</u> Y' <u>separates points in Y</u> if and only if for any $y_1 \neq y_2$ in Y, there is a $f\epsilon Y'$ such that $f(y_1) \neq f(y_2)$.

If Y' separates points in Y, then surely λ is one-to-one on Y onto $R(\lambda)$. Conversely, if λ is one-to-one on Y then Y' separates points in Y. Therefore, if Y' separates points in Y, then $\hat{I}$ is point-valued in Y on $D_o(\hat{I})$, and if Y' does not separate points in Y $\hat{I}$ may not be point-valued in Y on $D_o(\hat{I})$. The case in which Y' does not separate points in Y can be interesting; in Chapter IV we will show that the extension of the Lebesgue integral is coset-valued in Y when Y is a group, Z is the set of real numbers, and Y' is the set of all homomorphisms on Y into Z.

Assume that $\emptyset \neq Y_1' \subset Y_2' \subset (Y,Z)$, and let $\hat{I}_1$, $\hat{I}_2$ be the respective extensions of I by Y_1' and Y_2'. The following relationships show the dependence of $\hat{I}$ on Y': (1) $D(\hat{I}_2) \subset D(\hat{I}_1)$, (2) $D_o(\hat{I}_2) \subset D_o(\hat{I}_1)$, (3) if $\phi \epsilon D(\hat{I}_2)$, the restriction of $\hat{I}_2(\phi)$ to Y_1' is $\hat{I}_1(\phi)$, and (4) if $\phi \epsilon D_o(\hat{I}_2)$, then $\hat{I}_2(\phi) \subset \hat{I}_1(\phi)$.

We therefore see that the properties of $\hat{I}$ will depend on Y'. The general problem of extending I to functions mapping X into Y may be summarized as follows: no matter what I and Y we are given, select a $Y' \subset (Y,Z)$ so that as many as possible of the following conditions are satisfied: (1) $D(\hat{I})$ is non-empty, (2) $\hat{I} = I$ whenever Y=Z, (3) $D_o(\hat{I}) = D(\hat{I}) \neq \emptyset$ and Y' separates points in Y, and (4) $\hat{I}$ inherits most of the important properties of I. The second condition above would justify our calling $\hat{I}$ an extension of I, and the third condition says that $\hat{I}$ is point-valued in Y on its domain. Some of these conditions will be discussed in the next section and again in Chapters IV, V, and VI for various choices of X, Y, Z, I, and Y'.

One possible objection to extending I in this manner is that usually the values of $\hat{I}$ are in (Y',Z) and are not in Y. But, since we identify F_y and $\lambda^{-1}(F_y)$ we may think of (Y',Z) as an extension of the space Y. Then the values of $\hat{I}$ are in Y or the values are in the extension of Y. After all, the value of the Lebesgue integral of a finite measurable function can be in the extended real numbers.

Relation of $\hat{I}$ to the Pettis and Dunford Integrals

We will first discuss how $\hat{I}$ is related to Dunford's integral. Let A be a σ-algebra of subsets of X, μ a finite measure on A, Y a Banach space over the reals, Z the set of real numbers, $\bar{Y}$ the set of all continuous linear maps of Y into Z, Γ a closed linear manifold in $\bar{Y}$, and $\bar{\bar{Y}}(\bar{\Gamma})$ the set of all continuous linear maps of $\bar{Y}(\Gamma)$ into Z. $\bar{Y}$ and $\bar{\bar{Y}}$ have the usual norm topology. $I(\cdot)$ is the Lebesgue integral of measurable real-valued ψ over X with respect to μ. The domain of I is $D(I) = \{\psi \,\big|\, |I(\psi)| < +\infty\}$. $\hat{I}$ is the Y'-extension of I corresponding to $Y' \subset (Y,Z)$.

For each Γ we will take $D(Du,\Gamma) = \{\phi\epsilon(X,Y)|f(\phi)\epsilon D(I)$ for every f in $\Gamma\}$ as the domain of Dunford's integral. If $\phi\epsilon D(Du,\Gamma)$ and $F_\phi(f) = I[f(\phi)]$ for every f in Γ, then Dunford's integral of ϕ over X is defined to be F_ϕ (see Definition 2 of Chapter II). We will write $Du(\phi)$ for Dunford's integral of ϕ over X. Dunford showed that $Du(\phi)\epsilon\bar{\Gamma}$ for each ϕ in $D(Du,\Gamma)$ ([3], p.334).

In order to simplify the discussion here we have simplified Dunford's integral. Dunford has μ a finite signed measure and we have μ a finite measure. The domain of Dunford's integral (see Definition 2 of Chapter II) is $L_p(E)[Y,\Gamma]$ for $1 \le p \le +\infty$ and $E\epsilon A$. We have chosen $D(Du,\Gamma) = L_1(X)[Y,\Gamma]$ as the domain. But notice that $L_p(X)[Y,\Gamma] \subset D(Du,\Gamma)$ for $1 < p \le +\infty$ because μ is finite.

Let CH(Y,Z) be the set of all continuous homomorphisms on Y (thinking of the vectors of Y as a group) into Z (using addition in Z). If A is a subset of $\bar{Y}$, then $\ell(A)$ is the linear span of A in $\bar{Y}$ and $\overline{\ell(A)}$

is the closure of $\ell(A)$ in $\bar{Y}$. Finally, if H is a function on M and C is a subset of M, then $H|C$ is the restriction of H to C.

Theorem 1

(a) If $Y' = \Gamma$, then $Du(\phi) = \hat{I}(\phi)$ on $D(Du,\Gamma) = D(\hat{I})$.

(b) If $Y' \subset \Gamma$, then $D(Du,\Gamma) \subset D(\hat{I})$ and $Du(\phi)|Y' = \hat{I}(\phi)$ on $D(Du,\Gamma)$.

(c) If $Y' \subset CH(Y,Z)$, then $Y' \subset \bar{Y}$.

(d) Let $Y' \subset CH(Y,Z)$ and $\Gamma = \overline{\ell(Y')}$. If $G \epsilon \bar{\Gamma}$ and $G|Y' = \hat{I}(\phi)$ for some ϕ in $D(Du,\Gamma)$, then $G = Du(\phi)$.

Proof.

(a) and (b) follow directly from the definitions of $Du(\cdot)$ and $\hat{I}(\cdot)$.

(c) Let f be in Y'. It is easy to see that $f(ry) = rf(y)$ for each rational r and each y in Y ([17], p.19). The continuity of f implies that $f(ay) = af(y)$ for all reals a and all y in Y. Hence f is in $\bar{Y}$.

(d) From (c), $Y' \subset \bar{Y}$. It follows that Γ is a closed linear manifold in $\bar{Y}$. By (b), $Du(\phi) = G$ on Y', because they both equal $\hat{I}(\phi)$ on Y'. It follows that $Du(\phi) = G$ on $\ell(Y')$ because $\ell(Y') \subset \Gamma$ and both are linear on Γ into Z. Hence $Du(\phi) = G$ on Γ because they agree on $\ell(Y')$ and they are continuous on $\overline{\ell(Y')} = \Gamma$ into Z. □

Remark on Theorem 1. Theorem 1 states that if $Y' \subset CH(Y,Z), \Gamma = \overline{\ell(Y')}$, and $\phi \epsilon D(Du,\Gamma)$, then $\hat{I}(\phi)$ can be uniquely extended to a G in $\bar{\Gamma}$ and G must be $Du(\phi)$.

Now we will discuss how $\hat{I}$ is related to Pettis' integral defined in Definition 1 of Chapter II. Let $I(\psi,E)$ be the Lebesgue integral of measurable real-valued ψ over $E\epsilon A$ with respect to μ. The domain of $I(\cdot,E)$ is $D(I,E) = \{\psi\epsilon(X,Z)|\psi$ is measurable and $|I(\psi,E)| < +\infty\}$. $\hat{I}(\cdot,E)$ will be the Y'-extension of $I(\cdot,E)$ corresponding to Y' a subset of (Y,Z). We write $D(\hat{I},E)$ for the domain of $\hat{I}(\cdot,E)$ and $D_o(\hat{I},E) = \{\phi\epsilon D(\hat{I},E)|$ $\hat{I}(\phi,E)$ is in $R(\lambda)\}$. We now consider the values of $\hat{I}(\cdot,E)$ on $D_o(\hat{I},E)$ to be in Y. $P(\phi,E)$ denotes the Pettis integral of ϕ over E and $D(P,E) =$ $\{\phi\epsilon(X,Y)|\phi$ is Pettis integrable over $E\}$.

Theorem 2

(a) If $Y' = \bar{Y}$, then $D_o(\hat{I},E) = D(P,E)$ and $P(\phi,E) = \hat{I}(\phi,E)$ on $D_o(\hat{I},E)$.

(b) If $Y' \subset CH(Y,Z)$ and $\phi\epsilon D(P,E)$, then $\phi\epsilon D_o(\hat{I},E)$ and $P(\phi,E)\epsilon\hat{I}(\phi,E)$. If Y' separates points in Y, then $P(\phi,E) = \hat{I}(\phi,E)$.

(c) If $Y' = \bar{Y}$, then ϕ is Pettis integrable if and only if $\phi\epsilon \cap \{D_o(\hat{I},E)|E\epsilon A\}$.

Proof

(a) and (c) follow directly from the definitions (see Definition 1 of Chapter II).

(b) From (c) in Theorem 1 we have $Y' \subset \bar{Y}$. If ϕ is Pettis integrable over E, then there is a y_E in Y such that $f(y_E) = I(f(\phi),E)$ for every f in $\bar{Y}$. Therefore the same holds for each f in Y' and $y_E\epsilon\hat{I}(\phi,E)$. Hence $\phi\epsilon D_o(\hat{I},E)$ and $P(\phi,E)\epsilon\hat{I}(\phi,E)$. If Y' separates points in Y', then $\hat{I}(\phi,E)$ is a point in Y and $P(\phi,E) = \hat{I}(\phi,E)$. □

Properties of the Extension

Throughout this section, unless stated otherwise, X, Y, and Z are arbitrary non-empty sets, Y' is a non-empty subset of (Y,Z), and I is a function defined on non-empty $D(I) \subset (X,Z)$ with values in Z.

The basic properties of $\hat{I}$ that we will discuss here are: (1) continuity, (2) $\hat{I}$ reduces to I whenever Y=Z, (3) the composition theorem $T(\hat{I}(\phi)) = \hat{I}(T(\phi))$ and how this characterizes $\hat{I}$, and (4) that every $I':D(I') \subset (X,Y) \to (Y',Z)$ or Y is an $\hat{I}$ for some I. We will first discuss the continuity properties of $\hat{I}$.

Theorem 3. Let Y and Z be topological spaces.

(a) If I is continuous, then so is $\hat{I}(\phi)$ for every $\phi \epsilon D(\hat{I})$.

(b) If I is continuous and the members of Y' are continuous, then so is $\hat{I}$.

Proof. Recall that we have assumed that all function spaces have the product topology. We will write $y_\beta, f_\beta, \phi_\beta, \cdots$ for nets in $Y, Y', (X,Y), \cdots$ and usually omit any mention of the directed sets.

(a) If $\phi \epsilon D(\hat{I})$, then $\hat{I}(\phi) \epsilon (Y',Z)$. If $f_\beta \to f$ in Y' we must show $(\hat{I}(\phi))(f_\beta) \to (\hat{I}(\phi))(f)$. But $(\hat{I}(\phi))(f_\beta) = I(f_\beta(\phi))$ and $(\hat{I}(\phi))(f) = I(f(\phi))$. Let $\ell_\beta = f_\beta(\phi)$ and $\ell = f(\phi)$. Now $\ell_\beta \to \ell'$ for some $\ell' \epsilon D(I)$ if and only if $\ell_\beta(x) \to \ell'(x)$ for every $x \epsilon X$. But $\ell_\beta(x) = f_\beta(\phi(x)) \to f(\phi(x)) = \ell(x)$ for every $x \epsilon X$, so $\ell_\beta \to \ell = f(\phi)$. Also, ℓ_β and ℓ in D(I) implies that $I(\ell_\beta) \to I(\ell)$. But this means that $(\hat{I}(\phi))(f_\beta) = I(f_\beta(\phi)) = I(\ell_\beta) \to I(\ell) = I(f(\phi)) = (\hat{I}(\phi))(f)$. Hence $\hat{I}(\phi)$ is continuous for each $\phi \epsilon D(\hat{I})$.

(b) Let $\phi_\beta \to \phi$ in $D(\hat{I})$. We must show that $\hat{I}(\phi_\beta) \to \hat{I}(\phi)$. But since $\hat{I}(\phi_\beta)$ and $\hat{I}(\phi)$ are in (Y',Z), we must show that $\left(\hat{I}(\phi_\beta)\right)(f) \to \left(\hat{I}(\phi)\right)(f)$ for every $f\epsilon Y'$. Now $\left(\hat{I}(\phi_\beta)\right)(f) = I\left[f(\phi_\beta)\right]$ and $\left(\hat{I}(\phi)\right)(f) = I\left[f(\phi)\right]$ for every $f\epsilon Y'$. Hence it suffices to show that $I\left[f(\phi_\beta)\right] \to I\left[f(\phi)\right]$ for every $f\epsilon Y'$.

Let $\ell_\beta = f_o(\phi_\beta)$ and let $\ell = f_o(\phi)$ for $f_o \epsilon Y'$. Notice that ℓ_β and ℓ are in $D(I)$. Now $\phi_\beta \to \phi$ if and only if $\phi_\beta(x) \to \phi(x)$ for every $x\epsilon X$. But $\phi_\beta(x) \to \phi(x)$ for every $x\epsilon X$ implies that $f_o\left[\phi_\beta(x)\right] \to f_o\left[\phi(x)\right]$ for every $x\epsilon X$, because f_o is continuous. Therefore, $\ell_\beta \to \ell$ because $\ell_\beta(x) = f_o\left[\phi_\beta(x)\right] \to f_o\left[\phi(x)\right] = \ell(x)$ for every $x\epsilon X$. Hence $I(\ell_\beta) = I\left[f_o(\phi_\beta)\right] \to I\left[f_o(\phi)\right] = I(\ell)$, because I is continuous. Since f_o was arbitrary in Y', $I\left[f(\phi_\beta)\right] \to I\left[f(\phi)\right]$ for every $f\epsilon Y'$. □

Definition 4. If Z is a topological space, and if y_β and y are in Y, then we write $y_\beta \xrightarrow{w} y$ if and only if $f(y_\beta) \to f(y)$ for every $f\epsilon Y'$.

Remark on Definition 4. If Y is a linear topological vector space over the reals, Z the set of real numbers, and Y' all the continuous linear maps of Y into Z, then $y_\beta \xrightarrow{w} y$ is called weak convergence, or we say that y_β converges to y in the weak topology. It can happen that $y_\beta \xrightarrow{w} y$ and $y_\beta \not\to y$ ([18], p.174).

For the following corollary, recall that if $\phi\epsilon D_o(\hat{I})$, then $\hat{I}(\phi) = F_y \epsilon R(\lambda)$ and $\hat{I}(\phi) = \lambda^{-1}(F_y) = \{y\epsilon Y | f(y) = I\left[f(\phi)\right] \text{ for all } f\epsilon Y'\}$.

Corollary 1 to Theorem 3. Let Y and Z be topological spaces.

(a) $\hat{I}(\phi)$ is a continuous function on Y' into Z for every $\phi\epsilon D_o(\hat{I})$.

(b) If I is continuous and the members of Y' are continuous and net $\phi_\beta \epsilon D_o(\hat{I})$ with $\phi_\beta \to \phi \epsilon D_o(\hat{I})$, then for any $y_\beta \epsilon \hat{I}(\phi_\beta)$ and any $y \epsilon \hat{I}(\phi)$ we have $y_\beta \xrightarrow{w} y$.

Proof

(a) Let $\phi \epsilon D_o(\hat{I})$ and let $\hat{I}(\phi) = F_y$ for some $y \epsilon Y$. If $f_\beta \to f$ in Y', then $F_y(f_\beta) = f_\beta(y) \to f(y) = F_y(f)$. Hence $\hat{I}(\phi)$ is continuous.

(b) From Theorem 1 we have $f(y_\beta) = I[f(\phi_\beta)] \to I[f(\phi)] = f(y)$ for every $f \epsilon Y'$. Hence $y_\beta \xrightarrow{w} y$. □

We will now introduce the idea of a category for the next four theorems. For our purpose the following definition of a category $\mathcal{C}$ is sufficient.* Let $\mathcal{O}$ be a collection of non-empty sets. The elements of $\mathcal{O}$ are called the objects of $\mathcal{C}$. For each ordered pair of objects $<Y_1,Y_2>$ there is a set of functions, written $\hom(Y_1,Y_2)$, which are on Y_1 into Y_2. The members of $\hom(Y_1,Y_2)$ are called the morphisms of $\mathcal{C}$. The morphisms must satisfy the following two conditions: (a) if $<Y_1,Y_2,Y_3>$ is an ordered triple of objects and $f \epsilon \hom(Y_1,Y_2)$ and $g \epsilon \hom(Y_2,Y_3)$, then (their composition) $g(f) \epsilon \hom(Y_1,Y_3)$; and (b) if Y is an object, then the identity map $i_y \epsilon \hom(Y,Y)$. A category $\mathcal{C}$ consists of $\mathcal{O}$ together with the collection of morphisms. Note that $\mathcal{O}$ is not necessarily a set. If $\mathcal{O}$ is the collection of all non-empty sets, then $\hom(Y_1,Y_2)$ could be all functions on Y_1 into Y_2. $\mathcal{O}$, with these morphisms, is a category.

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*See Spanier ([19], p.14) for the more general definition of a category.

For the rest of this chapter $\mathcal{C}$ will be a category so that Z is an object and each hom(Y,Z), Y an object in $\mathcal{C}$, is non-empty.* We will now give four examples of this type of category.

Example 1. If Z is a non-empty set, then $\mathcal{O}$ is the collection of all non-empty sets and $\text{hom}(Y_1,Y_2) = (Y_1,Y_2)$.

Example 2. If Z is a topological space, then $\mathcal{O}$ is the collection of all topological spaces and $\text{hom}(Y_1,Y_2)$ is the set of all continuous maps on Y_1 into Y_2.

Example 3. If Z is the set of real numbers or the set of complex numbers under addition or multiplication, then $\mathcal{O}$ is the collection of all groupoids (sets where addition of elements is defined) and $\text{hom}(Y_1,Y_2)$ is the set of all homomorphisms on Y_1 into Y_2.

Example 4. If Z is the set of real numbers with the ordering $\leq$, then $\mathcal{O}$ is the collection of all partially ordered sets and $\text{hom}(Y_1,Y_2)$ is the set of all order preserving maps on Y_1 into Y_2.

If Y_1 and Y_2 are objects in $\mathcal{C}$, then for i=1,2 we write $Y_i' = \text{hom}(Y_i,Z)$, f_i for points in Y_i', F_i for elements of (Y_i',Z), ϕ_i for functions on X into Y_i, λ_i for the canonical map of Y_i into (Y_i',Z), $D(\hat{I},Y_i)$ for the domain of $\hat{I}$ corresponding to Y_i', and $D_o(\hat{I},Y_i)$ for $\{\phi_i \epsilon D(\hat{I},Y_i) | \hat{I}(\phi_i) \epsilon R(\lambda_i)\}$. If we are considering just one object Y we

*After these examples we will put one further restriction on $\mathcal{C}$.

will omit the subscripts on Y', f, F, etc.

Now we will put our final restriction on $\mathcal{C}$. We will assume that we are always working with a space X, a category $\mathcal{C}$, and an operator I, such that $D(\hat{I},Y)$ is non-empty for each object Y. For example, let $X = [0,1]$, Z be the set of real numbers, $\mathcal{C}$ the category of topological spaces and continuous functions (Example 2 above), I the Lebesgue integral over X, and $D(I) = \{\psi\epsilon(X,Z) \,\big|\, |I(\psi)| < +\infty\}$. Then every continuous map $\phi : X \to Y$ is in $D(\hat{I},Y)$ for each object Y in $\mathcal{C}$.

<u>Definition 5</u>. If $T\epsilon\hom(Y_1,Y_2)$, where Y_1 and Y_2 are objects in $\mathcal{C}$, then we extend T to $\hat{T}:(Y_1',Z) \to (Y_2',Z)$ as follows: $\hat{T}(F_1) = F_2$, where $F_2(f_2) = F_1\big(f_2(T)\big)$ for every $f_2\epsilon Y_2'$.

<u>Theorem 4</u>. (Composition Theorem.) Let $\langle Y_1,Y_2\rangle$ be any ordered pair of objects such that $\hom(Y_1,Y_2) \neq \emptyset$. If $T\epsilon\hom(Y_1,Y_2)$, then:

(a) $T\big(D(\hat{I},Y_1)\big) \subset D(\hat{I},Y_2)$, and $\hat{T}\big(\hat{I}(\phi_1)\big) = \hat{I}\big(T(\phi_1)\big)$ on $D(\hat{I},Y_1)$;

(b) $T\big(D_o(\hat{I},Y_1)\big) \subset D_o(\hat{I},Y_2)$, and $T\big(\hat{I}(\phi_1)\big) \subset \hat{I}\big(T(\phi_1)\big)$ on $D_o(\hat{I},Y_1)$ (equality holds if Y_2' separates points in Y_2); and

(c) $\hat{T}(F_{y_1}) = F_{T(y_1)}$, for each $y_1\epsilon\hat{I}(\phi_1)$, for each ϕ_1 in $D_o(\hat{I},Y_1)$.

<u>Proof</u>

(a) Let $\phi_1\epsilon D(\hat{I},Y_1)$. Then for any $f_2\epsilon Y_2'$ we have $f_2\big(T(\phi_1)\big) = \big(f_2(T)\big)(\phi_1)\epsilon D(I)$, because $f_2(T)\epsilon Y_1'$. Hence $T(\phi_1)\epsilon D(\hat{I},Y_2)$.

Let $F_1 = \hat{I}(\phi_1)$, $F_2 = \hat{T}(F_1)$, and $F_2' = \hat{I}\big(T(\phi_1)\big)$. Then $F_2(f_2) = F_1\big(f_2(T)\big) = I\Big[\big(f_2(T)\big)(\phi_1)\Big] = I\Big[f_2\big(T(\phi_1)\big)\Big] = F_2'(f_2)$ for every $f_2\epsilon Y_2'$. Hence $F_2 = F_2'$, and $\hat{T}\big(\hat{I}(\phi_1)\big) = \hat{I}\big(T(\phi_1)\big)$.

(b) Let $\phi_1 \epsilon D_o(\hat{I},Y_1)$, and let $y_1 \epsilon \hat{I}(\phi_1)$. Then $f_1(y_1) = I[f_1(\phi_1)]$ for every $f_1 \epsilon Y_1'$. But $f_2(T) \epsilon Y_1'$ for every $f_2 \epsilon Y_2'$. Therefore $f_2[T(y_1)] = [f_2(T)](y_1) = I\Big[[f_2(T)](\phi_1)\Big] = I\Big[f_2[T(\phi_1)]\Big]$ for every $f_2 \epsilon Y_2'$. This implies that $T(\phi_1) \epsilon D_o(\hat{I},Y_2)$ and $T(y_1) \epsilon \hat{I}[T(\phi_1)]$. Hence $T[\hat{I}(\phi_1)] \subset \hat{I}[T(\phi_1)]$. If Y_2' separates points in Y_2, then $\hat{I}[T(\phi_1)]$ is a point in Y_2 and $T[\hat{I}(\phi_1)] = \hat{I}[T(\phi_1)]$.

(c) Definition 5 implies that $\hat{T}(F_{y_1}) = F_{T(y_1)}$ for any $y_1 \epsilon Y_1$. □

Remark on Theorem 4. Theorem 4 generalizes Theorem 2.2 of Pettis ([2], p.280) which says: if T is a continuous linear map of a Banach space Y_1 into a Banach space Y_2 and if ϕ is a Pettis integrable function on X into Y_1, then $T(\phi)$ is a Pettis integrable function on X into Y_2 and $P[T(\phi),E] = T[P(\phi,E)]$ for every $E \epsilon A$.

Let M be the collection of all $\{Y,Y'\}$ where Y is an object in C and $Y' = \hom(Y,Z)$. For each $\{Y,Y'\}$ in M assume that I' is an operator defined for functions on X into Y with values in (Y',Z). If the values of I' are originally points in Y use the canonical map λ to get the values of I' in (Y',Z). For each $\{Y,Y'\}$ in M we write $D(I',Y)$ for the domain of I', and we assume that $D(I',Y)$ is always non-empty.

Definition 6. We say that I' is faithful on $A \subset (X,Z)$ with respect to $\{U,U'\}$ in M if and only if $U \subset Z$ and for $\{U,U'\}$ we have:

(a) $\emptyset \neq A \subset D(I',U) \cap D(I)$;

(b) $I': A \to R(\lambda)$, and $\lambda^{-1}[I'(\phi)] = I(\phi)$ on A.

Remark on Definition 6. Except in Chapter V, we will always consider the set U in Definition 6 to be equal to Z. In Chapter V, where I is the Lebesgue integral and Z is the complex numbers under multiplication, we will discuss the faithfulness of $\hat{I}$ with respect to $\{U,U'\}$ for four different objects U. Three of these objects are subsets of Z but not equal to Z.

It is easy to see that if $\hat{I}$ is faithful on A with respect to $\{Z,Z'\}$, then $A \subset D_o(\hat{I},Z)$. The next theorem shows that the converse is also true. Recall that Z' separates points in Z because the identity map $i_z \epsilon Z' = \hom(Z,Z)$.

Theorem 5. Assume that $D_o(\hat{I},Z) \neq \emptyset$. Then $\hat{I}$ is faithful on $D_o(\hat{I},Z)$ with respect to $\{Z,Z'\}$.

Proof. Let $\phi \epsilon D_o(\hat{I},Z)$. Then there is one and only one $z\epsilon Z$ such that $I\big(f(\phi)\big) = f(z)$ for every $f\epsilon Z'$. By definition $\hat{I}(\phi) = z$. Since $i_z \epsilon Z'$ we have $\phi = i_z(\phi)\epsilon D(I)$ and $I(\phi) = I\big(i_z(\phi)\big) = i_z(z) = z = \hat{I}(\phi)$. Therefore, $D_o(\hat{I},Z) \subset D(I)$ and $I=\hat{I}$ on $D_o(\hat{I},Z)$. □

Example 5. This example shows that the set A on which $\hat{I}$ is faithful may be quite small if Y' is large.

Let $\mathbf{C}$ be the category of topological spaces and continuous functions (Example 2), Z the set of real numbers, X a closed interval of real numbers [a,b], I the Lebesgue integral over X, and $D(I) = \{\psi\epsilon(X,Z) \mid |I(\psi)| < +\infty\}$. Assume that $\hat{I}$ is faithful on $A \subset (X,Z)$ with respect to $\{Z,Z'\}$. Then $I\big(f(\phi)\big) = f\big(I(\phi)\big)$ for every continuous map f on Z into Z, and each ϕ in A.

If $b-a \neq 1$, then $\hat{I}$ is not faithful on any non-empty $A \subset (X,Z)$ with respect to $\{Z,Z'\}$. Let $f_1(x) = 1$ for every x in Z and let ϕ belong to A. Then $b-a = I\big(f_1(\phi)\big) \neq 1 = f_1\big(I(\phi)\big)$.

If $b-a = 1$, we will now show that the only functions in A are the a.e. constant functions. Surely each a.e. constant function belongs to A. Therefore, assume that ϕ is not an a.e. constant function and that $\phi \epsilon D(I) \cap D(\hat{I},Z)$. We will show that ϕ does not belong to A. Let $\alpha = I(\phi)$ and let $K_n = [\alpha - 1/n, \alpha + 1/n]$, $H_n = \phi^{-1}(K_n)$ for $n=1,2,3,\cdots$.

There is a positive integer n_o such that $\mu(H_{n_o}) < 1$. If $\mu(H_n) = 1$ for all n, then we can show that ϕ must be an a.e. constant function. $\mu(H_n) = 1$ for all n implies that $1 = \lim_n \mu(H_n) = \mu(\bigcap_{n=1}^{\infty} H_n)$ because H_n is a decreasing sequence of sets. It follows that $\phi(\bigcap_{n=1}^{\infty} H_n) = \alpha$ and $\mu(\{\bigcap_{n=1}^{\infty} H_n\}^c) = 0$. Therefore ϕ must be an a.e. constant function.

There is a $f_o \epsilon Z'$ such that $0 \leq f_o \leq 1$, $f_o(\{K_{n_o}\}^c) = 0$, and $f_o(\alpha) = 1$. Then $I\big(f_o(\phi)\big) \leq \mu(H_{n_o}) < 1 = f_o(\alpha) = f_o\big(I(\phi)\big)$. Hence $\phi \notin A$.

Our next theorem shows how Theorem 4 characterizes all faithful extensions. Recall that I' is any operator which maps $D(I',Y)$, a non-empty subset of (X,Y), into (Y',Z) for each object Y in $\mathcal{C}$. The following theorem shows that if I' has the composition property (see Theorem 4) and is faithful, then I' and $\hat{I}$ agree on certain subsets of $D(I',Y)$.

<u>Theorem 6</u>. Assume that I' has the following two properties:

(a) $T\big(D(I',Y_1)\big) \subset D(I',Y_2)$ and $\hat{T}\big(I'(\phi_1)\big) = I'\big(T(\phi_1)\big)$ on $D(I',Y_1)$ for each $T\epsilon\hom(Y_1,Y_2)$ for each ordered pair of objects $<Y_1,Y_2>$ whose $\hom(Y_1,Y_2)$ is non-empty;

(b) I' is faithful on $A \subset (X,Z)$ with respect to $\{Z,Z'\}$.

Then $I' = \hat{I}$ on $\hat{A}(Y) = \{\phi\epsilon D(I',Y) \mid f(\phi)\epsilon A$ for each $f\epsilon Y'\}$ for each object Y. If $A = D(I',Z)$, then $\hat{A}(Y) = D(I',Y)$ for each object Y.

Proof. Let Y be a fixed object in $\mathcal{C}$.

(a) We first show that $\hat{A}(Y) \subset D(\hat{I},Y)$. Let $\phi\epsilon\hat{A}(Y)$. Then $f(\phi)\epsilon A \subset D(I)$ for each $f\epsilon Y'$. Hence $\phi\epsilon D(\hat{I},Y)$.

(b) Let $\phi\epsilon\hat{A}(Y)$, $F' = I'(\phi)$, and $F = \hat{I}(\phi)$. Assume that $f_o\epsilon Y'$.

$I\big(f_o(\phi)\big) = \lambda^{-1}\Big[I'\big(f_o(\phi)\big)\Big]$ because $f_o(\phi)\epsilon A$ and I' is faithful on A with respect to $\{Z,Z'\}$. If $z_o = I\big(f_o(\phi)\big)$, then $I'\big(f_o(\phi)\big) = F_{z_o}\epsilon(Z',Z)$.

Using the composition property of I' we get $\hat{f}_o(F') = I'\big(f_o(\phi)\big)$. Definition 5 implies that $\hat{f}_o(F') = F_o\epsilon(Z',Z)$ where $F_o(g) = F'\big(g(f_o)\big)$ for every $g\epsilon Z'$. Therefore $F_o = F_{z_o}$.

Recall that the identity map i_z belongs to Z'. Therefore $F(f_o) = I\big(f_o(\phi)\big) = z_o = i_z(z_o) = F_{z_o}(i_z) = F'\big(i_z(f_o)\big) = F'(f_o)$. This implies that $F(f) = F'(f)$ for every $f\epsilon Y'$, because f_o was an arbitrary point in Y'. Hence $F = F'$ on $\hat{A}(Y)$.

(c) Now assume that $A = D(I',Y)$. Surely $\hat{A}(Y) \subset D(I',Y)$. Therefore let $\phi\epsilon D(I',Y)$. The composition property of I' implies that $f(\phi)\epsilon D(I',Z) = A$ for every $f\epsilon Y' = \hom(Y,Z)$. Hence $\phi\epsilon\hat{A}(Y)$. $\square$

Corollary 1 to Theorem 6. If for each object Y $I':D(I',Y) \subset (X,Y) \to Y$, then $I'(\phi)\epsilon\hat{I}(\phi)$ for each $\phi\epsilon\hat{A}(Y)$ for each object Y.

Proof. Let Y be a fixed object in $\mathcal{C}$. Let $\phi\epsilon\hat{A}(Y)$ and let $y = I'(\phi)$. Then $\lambda\big(I'(\phi)\big) = F_y\epsilon(Y',Z)$. Theorem 6 implies that $F_y = \hat{I}(\phi)$. Hence $I'(\phi) = y\epsilon\lambda^{-1}\big(\hat{I}(\phi)\big)$. $\square$

We will now consider the case where the values of I' are subsets of Y. Assume that for each {Y,Y'} in M, $\tilde{I}$ is an operator defined for functions on X into Y with values non-empty subsets of Y. For each {Y,Y'} in M we write $D(\tilde{I},Y)$ for the domain of $\tilde{I}$, and we assume that $D(\tilde{I},Y)$ is always non-empty. The following corollary to Theorem 6 shows that if $\tilde{I}$ has a composition property (see Theorem 4) and $\tilde{I}$ agrees with I on a subset of $D(\tilde{I},Z) \cap D(I)$, then $\tilde{I}(\phi) \subset \hat{I}(\phi)$ on certain subsets of $D(\tilde{I},Y)$.

Corollary 2 to Theorem 6. Assume that $\tilde{I}$ has the following two properties:

(a) $T\big(D(\tilde{I},Y_1)\big) \subset D(\tilde{I},Y_2)$ and $T\big(\tilde{I}(\phi_1)\big) \subset \tilde{I}\big(T(\phi_1)\big)$ on $D(\tilde{I},Y_1)$ for each $T\epsilon\text{hom}(Y_1,Y_2)$ for each ordered pair of objects $<Y_1,Y_2>$ whose $\text{hom}(Y_1,Y_2)$ is non-empty;

(b) $\tilde{I}(\phi) = I(\phi)$ on a $\emptyset \neq A \subset D(\tilde{I},Z) \cap D(I)$.

Then $\tilde{I}(\phi) \subset \hat{I}(\phi)$ for each ϕ in $\hat{A}(Y)$ for each object Y. If $A = D(\tilde{I},Z)$, then $\hat{A}(Y) = D(\tilde{I},Y)$ for each object Y.

Proof. Let Y be a fixed object in $\mathcal{C}$.

(a) Let $\phi \epsilon \hat{A}(Y)$. Then $f(\phi) \epsilon A \subset D(I)$ for every $f \epsilon Y'$. Hence $\phi \epsilon D(\hat{I},Y)$. Therefore $\hat{A}(Y) \subset D(\hat{I},Y)$.

(b) Let $\phi \epsilon \hat{A}(Y)$. The composition property of $\tilde{I}$ implies that $f[\tilde{I}(\phi)] \subset \tilde{I}[f(\phi)]$ for every f in $Y' = \hom(Y,Z)$. But $\tilde{I}[f(\phi)] = I[f(\phi)]$ for each f in Y', because $\tilde{I} = I$ on A and $f(\phi) \epsilon A$ for each $f \epsilon Y'$. Hence $f[\tilde{I}(\phi)] = I[f(\phi)]$ for every $f \epsilon Y'$. Therefore $\phi \epsilon D_o(\hat{I},Y)$ and each y in $\tilde{I}(\phi)$ belongs to $\hat{I}(\phi)$. That is, $\tilde{I}(\phi) \subset \hat{I}(\phi)$.

(c) Assume that $A = D(\tilde{I},Z)$. Surely $\hat{A}(Y) \subset D(\tilde{I},Y)$. So let $\phi \epsilon D(\tilde{I},Y)$. Then the composition property of $\tilde{I}$ implies that $f(\phi) \epsilon D(\tilde{I},Z)$ for every $f \epsilon Y' = \hom(Y,Z)$. Hence $\phi \epsilon \hat{A}(Y)$. $\square$

Remark on Theorem 6. Theorem 6 and its corollaries generalize paragraph 2.21 of Pettis ([2], p.281). A discussion of this paragraph is in the section titled Relations to Other Integrals in Chapter II.

Application of Theorem 6. Let $(X,\mathcal{A},\mu)$ be a measure space, Z the set of real numbers, $I(\psi)$ the Lebesgue integral of measurable real-valued ψ over X with respect to μ, $D(I) = \{\psi \mid |I(\psi)| < +\infty\}$, and $\hat{I}$ the Y'-extension of I.

Assume that for each object Y, S is an integration theory for functions on X into Y with values in (Y',Z). If S is faithful on $A \subset (X,Z)$ with respect to $\{Z,Z'\}$ and has the composition property of Theorem 6, then every S-integrable function $\phi: X \to Y$ in $\hat{A}(Y)$ is in $D(\hat{I},Y)$ and $S(\phi) = \hat{I}(\phi)$ for each object Y. If A is the set of all S-integrable functions on X into Z, then the S integration theory is contained in the $\hat{I}$ theory. If the values of S are points in Y, then

$S(\phi)\epsilon\hat{I}(\phi)$ (they are equal if Y' separates points in Y).

Now assume that the values of S are subject in Y^* for each object Y. If S reduces to I on some subset A of (X,Z) and has the composition property of Corollary 2 to Theorem 6, then every S-integrable function $\phi:X \to Y$ in $\hat{A}(Y)$ is in $D_o(\hat{I},Y)$ and $S(\phi) \subset \hat{I}(\phi)$ for each object Y.

In Theorems 4, 5, and 6 we were given spaces X and Z, an operator I, and a certain type of category $\mathcal{C}$. $\mathcal{C}$ had to have the following three properties: (a) Z was an object in $\mathcal{C}$; (b) hom $(Y,Z) \neq \emptyset$ for each object Y; and (c) $D(\hat{I},Y) \neq \emptyset$ for each object Y. The space Y was allowed to range over the collection of objects in $\mathcal{C}$ and Y' was always hom (Y,Z).

In the following theorem we are given spaces X and Z, an operator I', and a category $\mathcal{C}$. We construct an operator I so that $I' = \hat{I}$. In corollary 1 (2) to Theorem 7 we are given space X, operator I' (set-valued operator $\tilde{I}$), and a collection of non-empty sets $\mathcal{D}$. We construct a space Z, an operator I, and a category $\mathcal{C}'$ so that $I' = \hat{I}(\tilde{I}\subset\hat{I})$. Category $\mathcal{C}'$ will have the following properties: (a) Z is an object; (b) Y' = hom(Y,Z) is non-empty for each object Y; and (c) $D(\hat{I},Y)$ is non-empty for each object $Y\neq Z$. Therefore, because of condition (c) above, $\mathcal{C}'$ is not necessarily $\mathcal{C}$.

We first consider the case where the values of I' are in (Y',Z). The corollaries consider the case where the values of I' are in Y.

For the next theorem let I' be any operator which maps D(I',Y), a non-empty subset of (X,Y), into (Y',Z) for each object Y. If Y_1 and

*For example, Rickart's integral [7] is set-valued.

Y_2 are objects, then for i=1,2 we will write I'_{y_i} for I' if I' maps $D(I',Y_i)$ into (Y'_i,Z).

Theorem 7. Assume that for any pair of objects Y_1 and Y_2 $\left(I'_{y_1}(\phi)\right)(f) = \left(I'_{y_2}(\phi)\right)(f)$ whenever $\phi \in D(I',Y_i)$ and $f \in Y'_i$ for i=1,2. Then there is a non-empty set $D(I) \subset (X,Z)$ and a function $I:D(I) \to Z$ so that $\hat{I}(\phi) = I'(\phi)$ on $D(I',Y)$ for each object Y.

Proof. For each object Y define $D(I,Y) = \{f(\phi) | f \in Y' \text{ and } \phi \in D(I',Y)\}$. Let $D(I) = \cup\{D(I,Y) | Y \text{ an object in } \mathcal{C}\}$. D(I) is a set because $D(I) \subset (X,Z)$.*

We now write I'_y for I' if I' maps $D(I',Y)$ into (Y',Z). Define $I:D(I) \to Z$ as follows: if $f(\phi)$ belongs to D(I) and $f(\phi)$ is in D(I,Y) for some object Y, then $I\left(f(\phi)\right) = \left(I'_y(\phi)\right)(f)$. Recall that $I'_y(\phi)$ is in (Y',Z). If $f(\phi)$ belongs to both $D(I,Y_1)$ and $D(I,Y_2)$, then by hypothesis $\left(I'_{y_1}(\phi)\right)(f) = \left(I'_{y_2}(\phi)\right)(f)$. Therefore $I(\cdot)$ is well-defined.

Now let Y be a fixed object in $\mathcal{C}$. It follows from the definition of $\hat{I}$ that $D(I',Y) \subset D(\hat{I},Y)$. If ϕ belongs to $D(I',Y)$, then $\hat{I}(\phi) = F$, where $F(f) = I\left(f(\phi)\right)$ for every f in Y'. Hence $I'(\phi) = \hat{I}(\phi)$ on $D(I',Y)$. □

Now we will consider the case where the values of I' are points in Y. Let $\mathcal{D}$ be a collection of non-empty sets and let X be a fixed non-empty set.

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*If B is a set, A is a collection of objects, and $A \subset B$, then A is a set ([20], p.256).

Corollary 1 to Theorem 7. Let $I':D(I',Y) \subset (X,Y) \to Y$ for each Y in $\mathcal{D}$. Then there is a non-empty set Z, a non-empty set $D(I) \subset (X,Z)$, a function $I:D(I) \to Z$, and a category $\mathcal{C}'$ so that: (1) $I' = \hat{I}$ on $D(I',Y)$ for each Y in $\mathcal{D}$; and (2) each Y in $\mathcal{D}$ is an object in $\mathcal{C}'$,

Proof. Let $Z = \{0,1\}$. Let $\mathcal{C}'$ be the category whose objects are the members of $\mathcal{D}$ together with Z, and $\hom(Y_1,Y_2) = (Y_1,Y_2)$ for every ordered pair of objects $\langle Y_1,Y_2\rangle$.

For each Y in $\mathcal{D}$ define $D(I,Y) = \{f(\phi) \mid f \in Y' = \hom(Y,Z)$ and $\phi \in D(I',Y)\}$. Let $D(I) = \cup\{D(I,Y) \mid Y \in \mathcal{D}\}$. We will write I'_y for I' if I' maps $D(I',Y)$ into Y. Define $I:D(I) \to Z$ as follows: if $f(\phi)$ is in $D(I)$ and $f(\phi)$ belongs to $D(I,Y)$ for some Y in $\mathcal{D}$, then $I[f(\phi)] = f[I'_y(\phi)]$. If $f(\phi)$ belongs to both $D(I,Y_1)$ and $D(I,Y_2)$, then $Y_1 = Y_2 = D(f)$ because a function has but one domain. Hence $I(\cdot)$ is well-defined.

Now let Y be a fixed member of $\mathcal{D}$. If ϕ belongs to $D(I',Y)$, then $I[f(\phi)] = f[I'(\phi)]$ for every f in $Y' = \hom(Y,Z)$. Therefore ϕ belongs to $D_o(\hat{I},Y)$ and $I'(\phi) \in \hat{I}(\phi)$. It follows that $I'(\phi) = \hat{I}(\phi)$ because Y' separates points in Y. $\square$

Now we will consider the case where the values of I' are non-empty subsets of Y.

Corollary 2 to Theorem 7. For each Y in $\mathcal{D}$, assume that $\tilde{I}$ maps $D(\tilde{I},Y)$, a non-empty subset of (X,Y), into non-empty subsets of Y. Then there is a non-empty set Z, a non-empty set $D(I) \subset (X,Z)$, a function $I:D(I) \to Z$, and a category $\mathcal{C}'$ so that: (1) $\tilde{I}(\phi) \subset \hat{I}(\phi)$ on $D(\tilde{I},Y)$ for each Y in $\mathcal{D}$; and (2) each Y in $\mathcal{D}$ is an object in $\mathcal{C}'$.

Proof. Let $Z = \{0,1\}$. Let $\mathcal{C}'$ be the category whose objects are the members of $\mathcal{D}$ together with Z. If Y is an object not equal to Z, define $\hom(Y,Z) = \{f\epsilon(Y,Z)|f$ is constant on $\tilde{I}(\phi)$ for each ϕ in $D(\tilde{I},Y)\}$. Otherwise, define $\hom(Y_1,Y_2)$ to be the set of all constant functions on Y_1 into Y_2 together with the identity map if $Y_1 = Y_2$. Each constant function on Y into Z belongs to $\hom(Y,Z)$ and therefore each $\hom(Y,Z)$ is non-empty. These objects and morphisms determine the category $\mathcal{C}'$. Define $D(I)$ as in Corollary 1 to Theorem 7.

For each $Y\epsilon\mathcal{D}$ and for each ϕ in $D(\tilde{I},Y)$, choose $y_\phi\epsilon\tilde{I}(\phi)$.* Define $I:D(I) \to Z$ as follows: if $f(\phi)$ is in $D(I)$ and $f(\phi)$ belongs to $D(I,Y)$ for some Y in $\mathcal{D}$, then $I\big(f(\phi)\big) = f(y_\phi)$.

Now let Y be a fixed member of $\mathcal{D}$. If $\phi\epsilon D(\tilde{I},Y)$, then $I\big(f(\phi)\big) = f(y_\phi)$ for every $f\epsilon Y' = \hom(Y,Z)$. Therefore $\phi\epsilon D_o(\hat{I},Y)$ and $y_\phi\epsilon\hat{I}(\phi)$. If $y'\epsilon\tilde{I}(\phi)$, then $f(y') = f(y_\phi)$ for every f in $Y' = \hom(Y,Z)$. Hence $\tilde{I}(\phi) \subset \hat{I}(\phi)$. □

Remark on Corollary 2 to Theorem 7. If for some Y_1 in $\mathcal{D}$ the collection $\{\tilde{I}(\phi)|\phi\epsilon D(\tilde{I},Y_1)\}$ is mutually disjoint, then the corollary shows that $\tilde{I} = \hat{I}$ on $D(\tilde{I},Y_1)$.

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*We are using a class form of the axiom of choice ([20], p.273). If $\mathcal{D}$ was a set, then we could use a set form of the axiom of choice ([20], p.33).

CHAPTER IV

THE REALS UNDER ADDITION AS THE RANGE SPACE OF THE FUNCTIONALS

In this chapter we extend the results of Chapter III by letting the operator I be the Lebesgue integral and the reals be the range space of the functionals. In Chapter III we used the symbol Z for the range space of the functionals, and now this space, the reals under addition, will be denoted by R.

Throughout this chapter $(X,\mathcal{A},\mu)$ will be a measure space with $\mathcal{A}$ a σ-algebra of subsets of X and μ an extended real-valued measure on $\mathcal{A}$. We always assume that μ is a complete measure and that $\mu(X) > 0$. $I(\psi,E)$ denotes the Lebesgue integral of ψ over $E\epsilon\mathcal{A}$ with respect to μ. For each $E\epsilon\mathcal{A}$, the domain of $I(\cdot,E)$ will be $D(I,E) = \{\psi\epsilon(E,R) \mid |I(\psi,E)| < +\infty\}$, and we write $D(I)$ for $D(I,X)$.

We will extend I to an operator $\hat{I}$ which is defined for functions mapping X into a (new) space Y. The properties of the space Y are used to organize this chapter into six sections. In the first section we start with Y an arbitrary non-empty set, add more and more properties to Y as we go from section to section, and in the last section Y is a partially ordered group topological space. In each section we derive the appropriate linear, order, or convergence properties of $\hat{I}$.

The set of functionals, Y', is always a non-empty subset of (Y,R). In general, the functionals will depend on the properties of the space

Y. If Y has a topology, then the functionals will be continuous maps of Y into R. The functionals will be homomorphisms whenever addition is defined on Y, and they will be order preserving maps if Y is a partially ordered set.

We will show that the extension $\hat{I}$ of the Lebesgue integral will inherit most of the elementary properties of the Lebesgue integral when Y has an addition, an ordering, and a topology defined on it. Except for Theorem 20, we never assume that there is any relation between the order, algebraic, and topological properties of Y (e.g. continuity of addition). When $\mathcal{C}$ is the category of groupoid topological spaces and continuous homomorphisms, we show that $\hat{I}$ is faithful on D(I) with respect to {R,R'} (see Definition 6 in Chapter III).

We never assume that Y' separates points in Y. This is particularly interesting when Y is a group, because then the values of $\hat{I}$ in Y are cosets in Y. If Y is a locally compact Abelian topological group and Y' is the set of all continuous homomorphisms on Y into R, then Hewitt and Ross ([21], p.390) give necessary and sufficient conditions on Y in order that Y' separate points in Y.

In the next chapter we take the complex numbers under multiplication as the range space of the functionals. In this chapter our interest is centered on extending the Lebesgue integral of a real-valued function to functions whose values lie in an abstract space Y, and so we chose R for the range space of the functionals. The complex numbers under multiplication is also a natural choice for the range space of the functionals because of Pontrjagin's duality theorem (Theorem 1 in

Chapter V). In Chapter V we extend the Lebesgue integral of a complex-valued function with respect to a complex measure to functions whose values lie in a topological group.

We assume all function spaces have the product topology whenever possible. Addition and $\leq$ are defined pointwise in (X,R), (Y,R), and (Y',R).

Y an Arbitrary Set

In this section Y is an arbitrary non-empty set and we will discuss convergence, linearity, absolute continuity, and other properties of $\hat{I}$. Recall that $Y' \subset (Y,R)$ and (canonical map) $\lambda(y) = F_y$ for every $y \in Y$, where $F_y(f) = f(y)$ for every $f \in Y'$. $R(\lambda)$ is the range of λ.

Definition 1. Let $E \in A$.

(a) $D(\hat{I},E) = \{\phi \in (E,Y) | f(\phi) \in D(I,E)$ for every $f \in Y'\}$, and $D(\hat{I}) = D(\hat{I},X)$.

(b) Let $\phi \in D(\hat{I},E)$. Define $F(f) = I\big(f(\phi),E\big)$ for every $f \in Y'$. If $F = F_y \in R(\lambda)$ for some $y \in Y$, then $\hat{I}(\phi,E) = F_y$ or $\lambda^{-1}(F_y)$. If $F \notin R(\lambda)$, then $\hat{I}(\phi,E) = F$. We write $\hat{I}(\phi) = \hat{I}(\phi,X)$.

(c) $D_o(\hat{I},E) = \{\phi \in D(\hat{I},E) | \hat{I}(\phi,E) \in R(\lambda)\}$, and $D_o(\hat{I}) = D_o(\hat{I},X)$.

Some of the immediate consequences of the above definition are:

(a) if $E_1 \subset E_2$, with both in A, then $D(\hat{I},E_2) \subset D(\hat{I},E_1)$,

(b) if $\phi_1 \in D(\hat{I},E)$, $\phi_2 \in (E,Y)$, $\phi_1 = \phi_2$ a.e. on E, then $\phi_2 \in D(\hat{I},E)$ and $\hat{I}(\phi_1,E) = \hat{I}(\phi_2,E)$,

(c) if $\phi_1 \in D_o(\hat{I},E)$, $\phi_2 \in (E,Y)$, $\phi_1 = \phi_2$ a.e. on E, then $\phi_2 \in D_o(\hat{I},E)$ and $\hat{I}(\phi_1,E) = \hat{I}(\phi_2,E)$,

(d) if $\mu(E) = 1$, then all constant functions in (E,Y) are in $D_o(\hat{I},E)$,

(e) if $\mu(E) < +\infty$, then all simple functions in (E,Y) are in $D(\hat{I},E)$, and

(f) if $\mu(E) = +\infty$ and there is a $f_r \in Y'$, where $f_r(y) = r$ for every $y \in Y$, then $D(\hat{I},E) = \emptyset$.

Convergence Theorems

The functionals enable us to use the topology of R and deduce the following convergence theorems.

Theorem 1. (Dominated Convergence Theorem.) If $\phi_n \in D(\hat{I},E)$ for $n=1,2,3,\cdots$, $\phi \in (E,Y)$, if $f(\phi_n)$ converges to $f(\phi)$ in measure on E for every $f \in Y'$, and if $|f(\phi_n)| \leq h$ a.e. on E for all n and all $f \in Y'$, with $|I(h,E)| < +\infty$, then $\phi \in D(\hat{I},E)$, and $\hat{I}(\phi_n,E) \to \hat{I}(\phi,E)$.

Proof. Let $F_n = \hat{I}(\phi_n,E)$ for $n=1,2,3,\cdots$. We must show that $\phi \in D(\hat{I},E)$ and $F_n(f) \to F(f)$ for every $f \in Y'$, where $F = \hat{I}(\phi,E)$, because convergence in (Y',R) is pointwise convergence.

For each fixed $f \in Y'$, we have, by the dominated convergence theorum for the Lebesgue integral, that $F_n(f) = I[f(\phi_n),E] \to I[f(\phi),E]$ and $|I[f(\phi),E]| < +\infty$. Hence $\phi \in D(\hat{I},E)$ and $F_n(f) \to F(f)$ for every f in Y'. □

Remark on Theorem 1. We will return to this theorem again in the next section when Y is a topological space and again when Y is a partially ordered set which is also a group. When Y is a topological space we will be able to consider $\phi_n \to \phi$ a.e. instead of convergence in measure.

There is a corollary to Theorem 1 for $D_o(\hat{I},E)$, but first we need the following definition. Recall that $y_n \xrightarrow{w} y$ in Y if and only if $f(y_n) \to f(y)$ for every f in Y'.

<u>Definition 2</u>. We say that Y is <u>weakly sequentially complete</u> if and only if given a sequence y_n of points in Y such that $\lim_n f(y_n)$ exists and is finite for every f in Y', then there is a y in Y such that $y_n \xrightarrow{w} y$.

<u>Remark on Definition 2</u>. If Y is a Banach space and Y' its dual, then Y is not necessarily weakly sequentially complete ([22], p.210).

<u>Corollary 1 to Theorem 1</u>. Assume now that Y is weakly sequentially complete and that $\phi_n \epsilon D_o(\hat{I},E)$ for $n=1,2,3,\cdots$. Then $\phi \epsilon D_o(\hat{I},E)$ and if $y_n \epsilon \hat{I}(\phi_n,E)$ for $n=1,2,3,\cdots$, and if $y \epsilon \hat{I}(\phi,E)$, then $y_n \xrightarrow{w} y$.

<u>Proof</u>. Let $y_n \epsilon \hat{I}(\phi_n,E)$ for $n=1,2,3,\cdots$. Then for each fixed $f \epsilon Y'$, we have, by the dominated convergence theorem for the Lebesgue integral, that $f(y_n) = I\left(f(\phi_n),E\right) \to I\left(f(\phi),E\right)$ and $\left|I\left(f(\phi),E\right)\right| < +\infty$. Hence, there is a $\bar{y} \epsilon Y$ so that $f(y_n) \to f(\bar{y})$ for every $f \epsilon Y'$. Therefore, for each $f \epsilon Y'$ we have $f(\bar{y}) = I\left(f(\phi),E\right)$, and so $\phi \epsilon D_o(\hat{I},E)$. If $y \epsilon \hat{I}(\phi,E)$, then $f(\bar{y}) = f(y)$ for every $f \epsilon Y'$, so we have $y_n \xrightarrow{w} y$ also. $\square$

<u>Theorem 2</u>. Let $\phi_n \epsilon D(\hat{I})$ for $n=1,2,3,\cdots$, $\phi \epsilon (X,Y)$, let $f(\phi_n)$ converge to $f(\phi)$ in measure for each $f \epsilon Y'$, and let $\lim_n \hat{I}(\phi_n,E)$ exist in (Y',R) for each $E \epsilon A$. Then $\phi \epsilon D(\hat{I})$, and $\hat{I}(\phi_n,E) \to \hat{I}(\phi,E)$ for each $E \epsilon A$.

Proof. Let $F_{n,E} = \hat{I}(\phi_n,E)$ for $n=1,2,3,\cdots$ and each $E\epsilon A$. By hypothesis, for each $E\epsilon A$ there is a $F_E\epsilon(Y',R)$ so that $F_{n,E}(f) \to F_E(f)$ for every $f\epsilon Y'$. This means that for each $E\epsilon A$ $F_{n,E}(f) = I\big(f(\phi_n),E\big) \to F_E(f)$ (finite) for each $f\epsilon Y'$. Therefore, for each fixed $f\epsilon Y'$, we have, by the Theorem in Appendix A, that $I\big(f(\phi_n),E\big) \to I\big(f(\phi),E\big)$ for every $E\epsilon A$ and $\big|I\big(f(\phi)\big)\big| < +\infty$. Hence $\phi\epsilon D(\hat{I})$ and $F_E = \hat{I}(\phi,E)$ for each $E\epsilon A$. It follows that $\hat{I}(\phi_n,E) = F_{n,E} \to F_E = \hat{I}(\phi,E)$ for every $E\epsilon A$. □

Remarks on Theorem 2. We will return to this theorem again in the next section when Y is a topological space. There we will consider $\phi_n \to \phi$ a.e. instead of convergence in measure. Theorem 2 generalizes Theorem 4.1 of Pettis ([2], p.290). Pettis' theorem says that if ϕ_n is a sequence of Pettis integrable functions converging weakly in measure to ϕ and if $\lim_n P(\phi_n,E)$ exists for every $E\epsilon A$, then ϕ is Pettis integrable and $P(\phi_n,E) \to P(\phi,E)$ for every $E\epsilon A$. $P(\cdot,E)$ is the Pettis integral over $E\epsilon A$ (see Definition 1 of Chapter II). Theorem 2 also generalizes theorem 58 of Dunford ([3], p.342) which we will not present here because it involves some special terminology which is not needed here.

Corollary 1 to Theorem 2. Assume now that Y is weakly sequentially complete and that $\phi_n\epsilon D_o(\hat{I},E)$ for all n and all E in A. Then $\phi\epsilon D_o(\hat{I},E)$ for all E in A and if $y_{n,E}\epsilon\hat{I}(\phi_n,E)$ for $n=1,2,\cdots$, and if $y_E\epsilon\hat{I}(\phi,E)$, then $y_{n,E} \xrightarrow{w} y_E$ for every E in A.

Proof. Now $\hat{I}(\phi_n,E) = F_{y_{n,E}}$ for each n and each $E\epsilon A$. By hypothesis, for each $E\epsilon A$ there is a $F_E\epsilon(Y',R)$ so that $f(y_{n,E}) = F_{y_{n,E}}(f) \to F_E(f)$

(finite) for each $f \in Y'$. Hence, for each $E \in A$ there is a $\bar{y}_E \in Y$ so that $y_{n,E} \xrightarrow{w} \bar{y}_E$.

On the other hand, since $F_{y_{n,E}}(f) = I(f(\phi_n), E)$, we have that $\lim_n I(f(\phi_n), E)$ exists and is finite for each $f \in Y'$ and each $E \in A$. Therefore, for each fixed $f \in Y'$, we have, by the Theorem in Appendix A, that $I(f(\phi_n), E) \to I(f(\phi), E)$ for every $E \in A$ and $|I(f(\phi))| < +\infty$. Hence, for each $f \in Y'$ and each $E \in A$, we have $f(\bar{y}_E) = I(f(\phi), E)$. This implies that $\phi \in D_o(\hat{I}, E)$ for each $E \in A$.

If $y_E \in \hat{I}(\phi, E)$ for each $E \in A$, then $f(\bar{y}_E) = f(y_E)$ for every $f \in Y'$ and each $E \in A$. Therefore $y_{n,E} \xrightarrow{w} y_E$ for each $E \in A$ also. □

Remarks on Corollary 1 to Theorem 2. This corollary generalizes theorem 4.1 of Pettis ([2], p.290) and theorem 62 of Dunford ([3], p.347). The requirement that each $\phi_n \in D_o(\hat{I}, E)$ for every $E \in A$ was necessary because it does not necessarily follow that $D_o(\hat{I}, E_2) \subset D_o(\hat{I}, E_1)$ whenever $E_1 \subset E_2$ ([2], p.302).

Linearity Properties

We again use the functionals to show that $\hat{I}(\phi, \cdot)$ is countably additive on A.

Definition 3. If $F_i \in (Y', R)$ for $i=1,2,3,\cdots$, then define $S_n = F_1 + F_2 + \cdots + F_n$ for $n=1,2,3,\cdots$. If there is a $F \in (Y', R)$ so that $S_n \to F$ (pointwise), then we write $\sum_{i=1}^{\infty} F_i = F$.

Theorem 3. Let $E_i \in A$ for $i=1,2,3,\cdots$ and $E_i \cap E_j = \emptyset$ whenever $i \neq j$. If $E = \bigcup_{i=1}^{\infty} E_i$ and $\phi \in D(\hat{I}, E)$, then $\hat{I}(\phi, E) = \sum_{i=1}^{\infty} \hat{I}(\phi, E_i)$.

Proof. Let $F_i = \hat{I}(\phi,E_i)$ for $i=1,2,3,\cdots$, and let $F = \hat{I}(\phi,E)$. Then for each $f\epsilon Y'$ we have ([17], p.201) that $S_n(f) = \sum_{i=1}^{n} F_i(f) = \sum_{i=1}^{n} I\big(f(\phi),E_i\big) \to I\big(f(\phi),E\big) = F(f)$. Hence $S_n \to F$ pointwise. □

Remarks on Theorem 3. Theorem 3 generalizes theorems 50, 53, and 54 of Dunford ([3], pp.339-341) where he discusses the complete additivity of his indefinite integral. Addition is required on Y in order to discuss the additivity of $\hat{I}(\phi,\cdot)$, whose values are in Y, for a $\phi\epsilon D_o(\hat{I},E)$. This, and the additivity of $\hat{I}(\cdot,E)$ on $D(\hat{I},E)$ and $D_o(\hat{I},E)$, will be studied in a later section when addition is defined on Y.

Absolute Continuity of $\hat{I}$

Definition 4. $B = \cap\{f^{-1}(0)|f\epsilon Y'\}$.

Remark on Definition 4. If Y is a group and the members of Y' are homomorphisms of Y into R, then surely B is non-empty. For now we will assume that B is non-empty.

Theorem 4. ($\hat{I}$ is absolutely continuous.) Let $E_i\epsilon A$ for $i=1,2,3,\cdots$, $E = \bigcup_{i=1}^{\infty} E_i$, and $\mu(E_i) \to 0$. Then if $\phi\epsilon D(\hat{I},E)$ we have $|\hat{I}(\phi,E_i)| \to 0$ in (Y',R), where $0(f) = 0$ for every $f\epsilon Y'$.

Proof. For any $F\epsilon(Y',R)$, $|F|(f) = |F(f)|$ for every $f\epsilon Y'$. Let $F_i = \hat{I}(\phi,E_i)$ for $i=1,2,3,\cdots$. For each $f\epsilon Y'$ we have ([23], p.124) $I\big(|f(\phi)|,E_i\big) \to 0$. But $|F_i|(f) = |F_i(f)| = |I\big(f(\phi),E_i\big)| \le I\big(|f(\phi)|,E_i\big) \to 0$ for each $f\epsilon Y'$. Hence $|F_i| \to 0$ pointwise. □

Corollary 1 to Theorem 4. Assume now that $\phi \epsilon D_o(\hat{I},E_i)$ for $i=1,2,3,\cdots$. If $y_i \epsilon \hat{I}(\phi,E_i)$ for $i=1,2,3,\cdots$, and $y \epsilon B$, then $y_i \xrightarrow{w} y$.

Proof. Theorem 4 implies that $f(y_i) = F_{y_i}(f) \to O(f) = 0$ for every $f \epsilon Y'$. But if $y \epsilon B$, then $f(y) = 0$ for every $f \epsilon Y'$. Hence $y_i \xrightarrow{w}$ each $y \epsilon B$. □

Remarks on Theorem 4 and Its Corollary. The corollary generalizes theorem 2.5 of Pettis ([2], p.283) and Theorem 4 generalizes theorems 50, 53, and 54 of Dunford ([3], pp.339-341). The theorem of Pettis states that $\|P(\phi,E_i)\| \to 0$ as $\mu(E_i) \to 0$ for a Pettis integrable function ϕ. When Y is a partially ordered set which is also a group, we will show that $\hat{I}(|\phi|,E_i) \to 0$ as $\mu(E_i) \to 0$.

Other Properties of $\hat{I}$

Definition 5. $\mathcal{W} = \{\lambda^{-1}(F_y) \mid F_y \epsilon R(\lambda)\}$.

Theorem 5. If $\mu(E) = 1$, then $\hat{I}(\cdot,E)$ maps $D_o(\hat{I},E)$ onto $\mathcal{W}$.

Proof. We know that $\hat{I}(\cdot,E)$ maps $D_o(\hat{I},E)$ into $\mathcal{W}$. Let $A \epsilon \mathcal{W}$. Then there is a $y \epsilon Y$ so that $A = \lambda^{-1}(F_y)$. Define $\phi_y(x) = y$ for every $x \epsilon E$. Surely $\phi_y \epsilon D(\hat{I},E)$. But $f(y) = I[f(\phi_y),E]$ for every $f \epsilon Y'$. Hence $\phi_y \epsilon D_o(\hat{I},E)$ and $A = \lambda^{-1}(F_y) = \hat{I}(\phi_y,E)$. □

Theorem 6. If $\phi \epsilon B$ a.e. on $E \epsilon \mathcal{A}$, then $\phi \epsilon D_o(\hat{I},E)$ and $\hat{I}(\phi,E) = B$.

Proof. For each $f \epsilon Y'$ we have $f(\phi) = 0$ a.e. on E. Therefore $I[f(\phi),E] = 0$ for every $f \epsilon Y'$. If $y \epsilon B$, then $f(y) = I[f(\phi),E]$ for every $f \epsilon Y'$. Hence $\phi \epsilon D_o(\hat{I},E)$ and $\hat{I}(\phi,E) = B$. □

Our next theorem and its corollary form a converse to Theorem 6.

Theorem 7. If Y' is countable and $\hat{I}(\phi,E) = B$ for every E in $\mathcal{A}$, then $\phi \epsilon B$ a.e. on X.

Proof. For each $f \epsilon Y'$ we have $I[f(\phi),E] = 0$ for every $E \epsilon \mathcal{A}$. This implies that $f(\phi) = 0$ a.e. on X for each $f \epsilon Y'$.

Let $Y' = \{f_1, f_2, f_3, \cdots\}$ and let $A_i = \phi^{-1}[f_i^{-1}(0)]$ for $i=1,2,3,\cdots$. Now $\bigcap_{i=1}^{\infty} A_i = \phi^{-1}\left(\bigcap_{i=1}^{\infty} f_i^{-1}(0)\right) = \phi^{-1}(B)$. But $\mu(A_i^c) = 0$ for each i. Hence $\mu\left[\left(\bigcap_{i=1}^{\infty} A_i\right)^c\right] = \mu\left(\bigcup_{i=1}^{\infty} A_i^c\right) = 0$. Therefore $\phi \epsilon B$ a.e. on X. □

Corollary 1 to Theorem 7. Let Y' be closed under addition and multiplication by real numbers. Then Y' is a linear vector space. Let H be a Hamel basis for Y'. If H is countable and $\hat{I}(\phi,E) = B$ for all E in $\mathcal{A}$, then $\phi \epsilon B$ a.e. on X.

Proof. Let $H = \{f_1, f_2, \cdots\}$ and let $B' = \cap\{f_i^{-1}(0) \mid f_i \epsilon H\}$. Then $B = B'$. Theorem 7 says that $\phi \epsilon B'$ a.e. on X. □

Theorem 7 and its corollary tell us when B is non-empty. If Y' is countable and $\hat{I}(\phi,E) = O\epsilon(Y',R)$ for all E in $\mathcal{A}$, then B is not empty. It would follow that $\mu(X) = 0$ if B was empty.

If we add that Y is weakly sequentially complete to the hypothesis of Corollary 1 to Theorem 4, then we can also show that B is not empty. The hypothesis of Corollary 1 to Theorem 4 implies that if $y_i \epsilon \hat{I}(\phi,E_i)$ for $i=1,2,3,\cdots$, then $f(y_i) \to 0$ for every $f \epsilon Y'$. If Y is weakly sequentially complete, then there is a $\bar{y} \epsilon Y$ so that $f(y_i) \to f(\bar{y})$ for every $f \epsilon Y'$. Hence $f(\bar{y}) = 0$ for every $f \epsilon Y'$ and $\bar{y} \epsilon B$.

Y a Topological Space

In this section Y will be an arbitrary topological space. Now we can discuss convergence in (X,Y) and we will restate Theorems 1 and 2 for $\phi_n \to \phi$ a.e. We will also look at three different ways of defining measurability for a $\phi: X \to Y$.

Definition 6. If A and K are topological spaces, then $C(A,K) = \{f \epsilon (A,K) | f \text{ is continuous}\}$.

Convergence Theorems

Theorem 8. (Dominated Convergence Theorem.) Assume the same hypothesis as that of Theorem 1 except now: (a) $Y' \subset C(Y,R)$, and (b) instead of $f(\phi_n)$ converging to $f(\phi)$ in measure on E for every f in Y' we assume that $\phi_n \to \phi$ a.e. on E. Then $\phi \epsilon D(\hat{I},E)$ and $\hat{I}(\phi_n,E) \to \hat{I}(\phi,E)$.

Proof. Since the members of Y' are continuous, we have $f(\phi_n) \to f(\phi)$ a.e. on E for each f in Y'. For each fixed f in Y' we may apply the dominated convergence theorem to the sequence $f(\phi_n)$. The rest of the proof is the same as that of Theorem 1. □

A corollary to Theorem 8 for $D_o(\hat{I},E)$ would be identical to Corollary 1 to Theorem 1 except that $Y' \subset C(Y,R)$ and $\phi_n \to \phi$ a.e. on E.

Theorem 9. Assume the same hypothesis as that of Theorem 2 except now: (a) μ is σ-finite, (b) $Y' \subset C(Y,R)$, and (c) instead of $f(\phi_n)$ converging to $f(\phi)$ in measure for every f in Y' we assume that $\phi_n \to \phi$ a.e. Then $\phi \epsilon D(\hat{I})$ and $\hat{I}(\phi_n,E) \to \hat{I}(\phi,E)$ for each $E \epsilon A$.

Proof. Since the members of Y' are continuous, we have $f(\phi_n) \to f(\phi)$ a.e. for each f in Y'. For each fixed f in Y', we may apply the Theorem in Appendix A to the sequence $f(\phi_n)$. The measure space is required to be σ-finite for a.e. convergence. The rest of the proof is identical to that of Theorem 2. □

A corollary to Theorem 9 for $D_o(\hat{I},E)$ would be the same as Corollary 1 to Theorem 2 except that $Y' \subset C(Y,R)$, μ is σ-finite, and $\phi_n \to \phi$ a.e.

The Domain of $\hat{I}$

Let $\mathcal{B}$ be the minimal σ-algebra over the open sets in Y.

Definition 7. Let $E \in \mathcal{A}$.

(a) $\phi \in (E,Y)$ is weakly measurable if and only if $f(\phi)$ is measurable for every f in Y'.

(b) $\phi \in (E,Y)$ is $(\mathcal{A},\mathcal{B})$-measurable if and only if $\phi^{-1}(\mathcal{B}) \subset \mathcal{A}$.

(c) $\phi \in (E,Y)$ is strongly measurable if and only if ϕ is the a.e. limit of a sequence of simple functions.

In general, these three definitions of measurability are different when $Y \neq R$. Some immediate consequences of the above definition are:

(a) if $Y' \subset C(Y,R)$, then strongly measurable functions are weakly measurable;

(b) if $Y' \subset C(Y,R)$, then $(\mathcal{A},\mathcal{B})$-measurable functions are weakly measurable; and

(c) if each f in Y' is bounded [e.g. if Y is compact and $Y' \subset C(Y,R)$] and $\mu(E) < +\infty$, then every weakly measurable function in

(E,Y) is in $D(\hat{I},E)$.

For a discussion of these definitions of measurability when Y is a Banach space see Johnson ([24], pp.39-41) and Pettis ([2], pp.278-279).

Y a Partially Ordered Set

In this section Y will be a partially ordered set with or without a topology.

Definition 8

(a) A relation $\leq$ is said to partially order Y (Y is a p.o. set) if and only if whenever $y_1,y_2,y_3 \epsilon Y$, $y_1 \leq y_2$, and $y_2 \leq y_3$, then $y_1 \leq y_3$.

(b) If Y is a p.o. set, we write $y_1 < y_2$ for any $y_1,y_2 \epsilon Y$ if and only if $y_1 \leq y_2$ and $y_1 \neq y_2$.

(c) The p.o. set Y is said to be fully ordered* (Y is a f.o. set) if and only if for every y_1 and y_2 in Y we have: (1) if $y_1 \leq y_2$ and $y_2 \leq y_1$, then $y_1 = y_2$, and (2) either $y_1 < y_2$, or $y_1 = y_2$, or $y_2 < y_1$.

(d) If Y is a p.o. set and Y is a topological space, we say that Y is a p.o. topological space.

We do not assume that there is any relation between the ordering and topology on Y when Y is a p.o. topological space. We now define the types of functionals we will be using when Y is a p.o. set.

Definition 9

(a) If Y is a p.o. set, then $O(Y,R)$ is the set of all order

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*Or linearly ordered, or simply ordered, or totally ordered.

preserving maps on Y into R.

(b) If Y is a p.o. topological space, then CO(Y,R) is the set of all continuous order preserving maps on Y into R.

We note that these sets of functionals in the above definition are non-empty because the function which is identically zero on Y belongs to both sets.

If Y is a p.o. set we make (X,Y) into a p.o. set as follows: if ϕ_1 and ϕ_2 are in (X,Y), then $\phi_1 \leq \phi_2$ if and only if $\phi_1(x) \leq \phi_2(x)$ for every $x \in X$. Similarly we define $\leq$ for (X,R), (Y,R), and (Y',R).

Theorem 10. (Monotone Convergence Theorem.) Let Y be a p.o. topological space, $Y' = CO(Y,R)$, $\phi_i \in (X,Y)$ for $i=1,2,\cdots$, $\phi_1 \leq \phi_2 \leq \phi_3 \leq \cdots \to \phi$, ϕ_1 and $\phi \in D(\hat{I})$,* and let each ϕ_i be weakly measurable for $i \geq 2$. Then $\hat{I}(\phi_1) \leq \hat{I}(\phi_2) \leq \hat{I}(\phi_3) \leq \cdots \to \hat{I}(\phi)$.

Proof. For each $f \in Y'$ we have $f(\phi_1) \leq f(\phi_2) \leq \cdots \to f(\phi)$. Since $\phi_1, \phi \in D(\hat{I})$ we have that each $\phi_i \in D(\hat{I})$ for $i \geq 2$. Let $F_i = \hat{I}(\phi_i)$ for $i=1,2,3,\cdots$, and let $F = \hat{I}(\phi)$. Therefore, for each fixed $f \in Y'$, we have, by the monotone convergence theorem for the Lebesgue integral, that $F_1(f) = I[f(\phi_1)] \leq F_2(f) = I[f(\phi_2)] \leq \cdots \to I[f(\phi)] = F(f)$. Hence $F_1 \leq F_2 \leq \cdots \to F$ (pointwise). □

We will discuss a corollary to Theorem 10 for $D_o(\hat{I})$ when Y is

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*We could omit the condition that $\phi \in D(\hat{I})$ if the extended real numbers was used as the range space of the functionals.

a partially ordered set which is also a group.

Definition 10. Let Y be a p.o. set and let $A_1, A_2 \epsilon \{\lambda^{-1}(F_y) | y \epsilon Y\}$. Then $A_1 \leq A_2$ if and only if $A_1 = A_2$ or $y_1 < y_2$ for every $y_i \epsilon A_i$, i=1,2.

Theorem 11. Let Y be a p.o. set and $Y' \subset O(Y,R)$.

(a) If $\phi_1, \phi_2 \epsilon D(\hat{I},E)$, $\phi \epsilon (E,Y)$ and is weakly measurable, and $\phi_1 \leq \phi \leq \phi_2$ a.e. on E, then $\phi \epsilon D(\hat{I},E)$ and $\hat{I}(\phi_1,E) \leq \hat{I}(\phi,E) \leq \hat{I}(\phi_2,E)$.

(b) If Y is a f.o. set, and $\phi_1 \leq \phi_2$ a.e. on E with $\phi_1, \phi_2 \epsilon D_o(\hat{I},E)$, then $\hat{I}(\phi_1,E) \leq \hat{I}(\phi_2,E)$ (as sets in Y).

Proof

(a) For each $f \epsilon Y'$ we have $f(\phi_1) \leq f(\phi) \leq f(\phi_2)$ a.e. on E and $|I[f(\phi_i),E]| < +\infty$ for i=1,2. The result follows by recalling that $\leq$ on (Y',R) was defined pointwise.

(b) Assume that $\hat{I}(\phi_1,E) \neq \hat{I}(\phi_2,E)$. This means that the two sets must be disjoint. Let $y_i \epsilon \hat{I}(\phi_i,E)$ for i=1,2. Since Y is a f.o. set we must have either $y_1 < y_2$, or $y_2 < y_1$, because $y_1 \neq y_2$. If $y_1 < y_2$ we are finished. Therefore assume that $y_2 < y_1$. We will show this implies that the two sets are not disjoint.

If $y_2 < y_1$, then $f(y_2) \leq f(y_1)$ for every $f \epsilon Y'$. Now $\phi_1 \leq \phi_2$ a.e. on E implies that $f(y_1) = I[f(\phi_1),E] \leq I[f(\phi_2),E] = f(y_2)$ for every $f \epsilon Y'$. Hence $f(y_1) = f(y_2)$ for every $f \epsilon Y'$. Therefore $y_2 \epsilon \lambda^{-1}(F_{y_1}) = \hat{I}(\phi_1,E)$ and the two sets are not disjoint. □

We will return to the order properties of $\hat{I}$ again when addition is defined on the p.o. set Y.

In order to develop the linearity properties of $\hat{I}$ fully, addition is required on Y. We will do this in the next section.

Y a Groupoid

In this section Y will be a set, with or without a topology, with addition defined on it. We will discuss the faithfulness and linearity properties of $\hat{I}$.

Definition 11

(a) Y is a groupoid (gpd) under addition $\oplus$ if and only if there is a $H: Y \times Y \to Y$ so that for every $y_1, y_2 \epsilon Y$, $y_1 \oplus y_2 = H(y_1, y_2)$.

(b) If Y is a gpd and Y is a topological space, then we say that Y is a gpd topological space.

We do not assume that addition on Y is a continuous operation. If Y is a gpd then so is (X,Y) under $(\phi_1 \oplus \phi_2)(x) = \phi_1(x) \oplus \phi_2(x)$ for every $x \epsilon X$. With addition on Y, we will further restrict the functionals we will use.

Definition 12

(a) If Y is a gpd, then H(Y,R) is the set of all homomorphisms on Y into R.

(b) If Y is a gpd topological space, then CH(Y,R) is the set of all continuous homomorphisms on Y into R.

These sets of functionals in the above definition are non-empty because the function which is identically zero on Y belongs to each set of functionals.

$\hat{I}$ is Faithful

Let $\mathcal{C}$ be a category where the objects are gpd topological spaces and the morphisms are continuous homomorphisms (see Chapter III for the definition of a category). That is, if Y_1 and Y_2 are objects in $\mathcal{C}$, then $\hom(Y_1,Y_2)$ is the set of all continuous homomorphisms on Y_1 into Y_2.

We will write $D(\hat{I},Y) \subset (X,Y)$ for the domain of $\hat{I}$ corresponding to the object Y. In Chapter III we assumed that the category $\mathcal{C}$ had the property that $D(\hat{I},Y)$ was non-empty for each object Y. If $\mu(X) < +\infty$, then all the constant functions on X into Y belong to $D(\hat{I},Y)$ for each object Y. Therefore, we will assume that $\mu(X) < +\infty$. For each object Y, $Y' = \hom(Y,R)$.

Theorem 12. $\hat{I}$ is faithful on $D(I) = D(\hat{I},R)$ with respect to $\{R,R'\}$.

Proof. See Definition 6 of Chapter III for the definition of $\hat{I}$ being faithful on $A \subset (X,R)$ with respect to $\{U,U'\}$.

(a) Let $f_r(y) = ry$ for every $y \in R$, for each $r \in R$. Now $R' = \hom(R,R) = CH(R,R)$. It is well known ([17], p.21) that $R' = \{f_r | r \in R\}$.

(b) We first show that $D(I) = D(\hat{I},R)$. Let r be a non-zero real number. If $\phi \in D(\hat{I},R)$, then $f_r(\phi) = r\phi \in D(I)$. Hence $\phi \in D(I)$ and $D(\hat{I},R) \subset D(I)$.

Let $\phi \in D(I)$. Then $f_r(\phi) = r\phi \in D(I)$ for every $r \in R$. Hence $\phi \in D(\hat{I},R)$ and $D(I) \subset D(\hat{I},R)$.

(c) Now we show that $\hat{I}: D(\hat{I},R) \to R(\lambda)$ and $\lambda^{-1}[\hat{I}(\phi)] = I(\phi)$ on $D(\hat{I},R)$. Let $\phi \in D(\hat{I},R)$. Then $I[f_r(\phi)] = r[I(\phi)]$ for every $f_r \in R'$. If

$y = I(\phi)$, then $I\left(f_r(\phi)\right) = F_y(f_r)$ for every $f_r \epsilon Y'$. Hence $\hat{I}(\phi) = F_y \epsilon R(\lambda)$. It follows that $\lambda^{-1}(\hat{I}(\phi)) = y = I(\phi)$ because R' separates points in R. $\square$

Linearity Properties of $\hat{I}$

Addition on Y will enable us to deduce that $\hat{I}$ is additive on $D(\hat{I})$. We will also see in what sense $\hat{I}$, with values in Y, is completely additive on A. The complete additivity of $\hat{I}$, with values in (Y',R), on A was discussed in the first section in this chapter.

Theorem 13. Assume that Y is a gpd, $Y' \subset H(Y,R)$, and that $\phi_1, \phi_2 \epsilon D(\hat{I},E)$. Then $\phi_1 \oplus \phi_2 \epsilon D(\hat{I},E)$ and $\hat{I}(\phi_1 \oplus \phi_2, E) = \hat{I}(\phi_1,E) + \hat{I}(\phi_2,E)$.

Proof. If $\phi_1, \phi_2 \epsilon D(\hat{I},E)$, then $f(\phi_1 \oplus \phi_2) = f(\phi_1) + f(\phi_2) \epsilon D(I,E)$ for every $f \epsilon Y'$, because $f(\phi_1)$ and $f(\phi_2)$ belong to $D(I,E)$ for every $f \epsilon Y'$. Hence $\phi_1 \oplus \phi_2 \epsilon D(\hat{I},E)$.

Let $F = \hat{I}(\phi_1 \oplus \phi_2, E)$ and let $F_i = \hat{I}(\phi_i, E)$ for i=1,2. Then, for each $f \epsilon Y'$, we have $(F_1+F_2)(f) = F_1(f) + F_2(f) = I\left(f(\phi_1) + f(\phi_2),E\right) = I\left(f(\phi_1 \oplus \phi_2),E\right) = F(f)$. Hence $F_1 + F_2 = F$. $\square$

Remark on Theorem 13. Theorem 13 says that $D(\hat{I},E)$ is a sub-groupoid of the groupoid (E,Y) and $\hat{I}(\cdot,E)$ is additive on $D(\hat{I},E)$ into (Y',R) whenever Y is a groupoid and $Y' \subset H(Y,R)$. We have the following corollary for $D_o(\hat{I},E)$.

Corollary 1 to Theorem 13. Assume that Y is a gpd, $Y' \subset H(Y,R)$, and that ϕ_1, ϕ_2 are in $D_o(\hat{I},E)$. Then $\phi_1 \oplus \phi_2$ is in $D_o(\hat{I},E)$ and $\hat{I}(\phi_1,E) \oplus \hat{I}(\phi_2,E) \subset \hat{I}(\phi_1 \oplus \phi_2,E)$.

Proof. Addition of sets in Y is defined as $A_1 \oplus A_2 = \{y_1 \oplus y_2 | y_1 \epsilon A_1$ and $y_2 \epsilon A_2\}$.

Let $y_i \epsilon \hat{I}(\phi_i, E)$ for i=1,2. Then, for each $f \epsilon Y'$, we have $f(y_1 \oplus y_2) = f(y_1) + f(y_2) = I[f(\phi_1), E] + I[f(\phi_2), E] = I[f(\phi_1 \oplus \phi_2), E]$. Hence $y_1 \oplus y_2$ is in $\hat{I}(\phi_1 \oplus \phi_2, E)$ and $\phi_1 \oplus \phi_2 \epsilon D_o(\hat{I}, E)$. □

Remark on Corollary 1 to Theorem 13. The corollary says that $D_o(\hat{I}, E)$ is a sub-groupoid of $D(\hat{I}, E)$ whenever Y is a groupoid and $Y' \subset H(Y, R)$. In a later section, when Y is a group, we will show that $\hat{I}(\cdot, E)$ is additive on $D_o(\hat{I}, E)$.

The next two theorems are actually corollaries to Theorem 3. In order to state Theorem 3, for the case where the values of $\hat{I}$ are in Y, addition is required on Y. Therefore, the theorems are in this section.

Theorem 14. Assume that $E_1, E_2 \epsilon \mathcal{A}$, $E_1 \cap E_2 = \emptyset$, Y is a gpd, $Y' \subset H(Y,R)$, and that $\phi \epsilon D_o(\hat{I}, E_i)$ for i=1,2. Then $\phi \epsilon D_o(\hat{I}, E_1 \cup E_2)$ and $\hat{I}(\phi, E_1) \oplus \hat{I}(\phi, E_2) \subset \hat{I}(\phi, E_1 \cup E_2)$.

Proof. Let $y_i \epsilon \hat{I}(\phi, E_i)$ for i=1,2. Then, for each $f \epsilon Y'$, we have $f(y_1 \oplus y_2) = f(y_1) + f(y_2) = I[f(\phi), E_1] + I[f(\phi), E_2] = I[f(\phi), E_1 \cup E_2]$. Therefore $y_1 \oplus y_2$ is in $\hat{I}(\phi, E_1 \cup E_2)$ and $\phi \epsilon D_o(\hat{I}, E_1 \cup E_2)$. □

In a later section, when Y is a group, we will show that $\hat{I}(\phi, E_1) \oplus \hat{I}(\phi, E_2) = \hat{I}(\phi, E_1 \cup E_2)$. We will extend Theorem 14 to countable collections of disjoint measurable sets but first we need the following definition. Recall that $y_n \xrightarrow{w} y$ if and only if $f(y_n) \to f(y)$ for every $f \epsilon Y'$.

Definition 13. Let $y_i \epsilon Y$ for $i=1,2,3,\cdots$, and let h map the positive integers into $\{0,1\}$. If $s_1 = y_1$, define s_n, $n \geq 2$, inductively as follows: $s_n = s_{n-1} \oplus y_n$ if $h(n) = 1$ and $s_n = y_n \oplus s_{n-1}$ if $h(n) = 0$. If there is a $y \epsilon Y$ so that $s_n \xrightarrow{w} y$, we write $h \sum_{i=1}^{n} y_i \xrightarrow{w} y$.

Theorem 15. Assume that Y is a gpd, $Y' \subset H(Y,R)$, Y is weakly sequentially complete, $E_i \epsilon A$ for $i=1,2,3,\cdots$, $E_i \cap E_j = \emptyset$ for $i \neq j$, $E = \bigcup_{i=1}^{\infty} E_i$, $\phi \epsilon D_o(\hat{I},E_i)$ for $i=1,2,3,\cdots$, and that $\phi \epsilon D(\hat{I},E)$. Then $\phi \epsilon D_o(\hat{I},E)$, and if $y_i \epsilon \hat{I}(\phi,E_i)$ for $i=1,2,3,\cdots$ and $y \epsilon \hat{I}(\phi,E)$, then $h \sum_{i=1}^{n} y_i \xrightarrow{w} y$ for every h.

Proof. Let h map the positive integers into $\{0,1\}$. For each fixed $f \epsilon Y'$ we have $f(s_n) = \sum_{i=1}^{n} I\big(f(\phi),E_i\big) \to I\big(f(\phi),E\big)$, because $\big|I\big(f(\phi),E\big)\big| < +\infty$ ([18], p.201). Therefore there is a $\bar{y} \epsilon Y$ so that $f(s_n) \to f(\bar{y})$ for every $f \epsilon Y'$. Hence $\bar{y} \epsilon \hat{I}(\phi,E)$ and $\phi \epsilon D_o(\hat{I},E)$.

If $y \epsilon \hat{I}(\phi,E)$, then $f(y) = f(\bar{y})$ for every $f \epsilon Y'$. Hence $s_n \xrightarrow{w} y$ also. □

Remark on Theorem 15. Theorem 15 generalizes Theorem 2.4 of Pettis ([2], p.283).

Y a Group

In this section Y is a group with identity θ. We will use $y_1 \oplus y_2$ and $y_1 - y_2$ for addition and subtraction in Y.

We will discuss linearity properties of $\hat{I}$, algebraic properties of the domain of $\hat{I}$, and some other properties of $\hat{I}$.

The set of functionals, Y', will always be a subset of $H(Y,R)$. Recall that $B = \cap\{f^{-1}(0) | f \epsilon Y'\}$. Then B is a normal subgroup of Y.

B is non-empty because θ belongs to B. We will write Y/B for $\{y\oplus B \mid y\epsilon Y\}$. Y/B is a disjoint decomposition of Y by (left) cosets of B. If we define $\oplus$ on Y/B as $(y_1\oplus B) \oplus (y_2\oplus B) = (y_1\oplus y_2) \oplus B$, then Y/B is a group with identity B. For each $\phi\epsilon D_o(\hat{I},E)$, $\hat{I}(\phi,E) = y\oplus B$ for some $y\epsilon Y$, because each $f\epsilon Y'$ is constant on each $y \oplus B$ in Y/B. That is, the value of $\hat{I}(\cdot,E)$, in Y, on $D_o(\hat{I},E)$ is a coset of B.

Linearity Properties

The following theorem improves the inequalities of Theorem 14 and of Corollary 1 to Theorem 13 to equalities.

Theorem 16

(a) If $\phi_1,\phi_2\epsilon D_o(\hat{I},E)$, then $\hat{I}(\phi_1,E) \oplus \hat{I}(\phi_2,E) = \hat{I}(\phi_1\oplus\phi_2,E)$.

(b) If $E_1,E_2\epsilon A$, $E_1 \cap E_2 = \emptyset$, and if $\phi\epsilon D_o(\hat{I},E_i)$ for i=1,2, then $\hat{I}(\phi,E_1) \oplus \hat{I}(\phi,E_2) = \hat{I}(\phi,E_1\cup E_2)$.

Proof

(a) Let $y_i\epsilon\hat{I}(\phi_i,E)$ for i=1,2. Corollary 1 to Theorem 13 implies that $y_1 \oplus y_2\epsilon\hat{I}(\phi_1\oplus\phi_2,E)$. Then $(y_1\oplus y_2) \oplus B = \hat{I}(\phi_1\oplus\phi_2,E)$. But $\hat{I}(\phi_1,E) \oplus \hat{I}(\phi_2,E) = (y_1\oplus B) \oplus (y_2\oplus B) = (y_1\oplus y_2) \oplus B$.

(b) Let $y_i\epsilon\hat{I}(\phi,E_i)$ for i=1,2. Theorem 14 implies that $y_1 \oplus y_2\epsilon\hat{I}(\phi,E_1\cup E_2)$. Hence $\hat{I}(\phi,E_1) \oplus \hat{I}(\phi,E_2) = (y_1\oplus B) \oplus (y_2\oplus B) = (y_1\oplus y_2) \oplus B = \hat{I}(\phi,E_1\cup E_2)$. □

Algebraic Properties of the Domain of $\hat{I}$

Definition 14

(a) $K(E) = \{\phi\epsilon D(\hat{I},E) \mid \hat{I}(\phi,E) = B\}$.

(b) If $E\epsilon A$, then $K_o(E) = \{\phi\epsilon(E,Y) \mid \phi\epsilon B$ a.e. on $E\}$.

$K(E)$ and $K_o(E)$ are both non-empty because the function which is identically equal to θ on E belongs to both sets.

<u>Theorem 17</u>

(a) $D(\hat{I},E)$ is a subgroup of the group (E,Y).

(b) $D_o(\hat{I},E)$ is a subgroup of $D(\hat{I},E)$.

(c) $K(E)$ is a normal subgroup of $D_o(\hat{I},E)$.

(d) $K_o(E)$ is a subgroup of $K(E)$.

<u>Proof</u>. We will use the following result from algebra. If G is a group and $H \subset G$, then H is a subgroup of G if and only if $a-b \in H$ whenever $a,b \in H$.

(a) Let $\phi_1,\phi_2 \in D(\hat{I},E)$. It follows that $\phi_1-\phi_2 \in D(\hat{I},E)$ because $f(\phi_1-\phi_2) = f(\phi_1) - f(\phi_2)$ for every $f \in Y'$.

(b) Let $\phi_1,\phi_2 \in D_o(\hat{I},E)$ and let $y_i \in \hat{I}(\phi_i,E)$ for $i=1,2$. Then $f(y_1-y_2) = f(y_1) - f(y_2) = I\left(f(\phi_1-\phi_2),E\right)$ for every $f \in Y'$. Hence $\phi_1-\phi_2 \in D_o(\hat{I},E)$.

(c) Theorem 16 implies that $\hat{I}(\cdot,E)$ is a homomorphism of $D_o(\hat{I},E)$ into Y/B. The kernel of the homomorphism is $K(E)$. Therefore $K(E)$ is a normal subgroup of $D_o(\hat{I},E)$.

(d) Surely $K_o(E) \subset K(E)$ (Theorem 6). If $\phi_1,\phi_2 \in K_o(E)$, then $\phi_1-\phi_2 \in K_o(E)$ because B is a group. Hence $K_o(E)$ is a subgroup of $K(E)$. □

We will write $D_o(\hat{I},E)/K(E)$ for $\{\phi \oplus K(E) \mid \phi \in D_o(\hat{I},E)\}$. Then $D_o(\hat{I},E)/K(E)$ is a disjoint decomposition of $D_o(\hat{I},E)$ by (left) cosets of $K(E)$.

Other Properties of $\hat{I}$

Theorem 18. Let $\mu(E) = 1$ and $\phi\epsilon(E,Y)$. Then $\phi\epsilon D_o(\hat{I},E)$ if and only if either

(a) there is a $y\epsilon Y$ so that $\phi\epsilon y\oplus B$ a.e. on E, or

(b) there is a $y\epsilon Y$ and a $\phi'\epsilon K(E)$, $\phi'\notin K_o(E)$, so that $\phi = y\oplus\phi'$ a.e. on E.

Proof

(1) If $\phi\epsilon y\oplus B$ a.e. on E, then $f(\phi) = f(y)$ a.e. on E for every $f\epsilon Y'$. Hence, ϕ is weakly measurable and $I\left(f(\phi),E\right) = f(y)$ for every f in Y'. Therefore ϕ belongs to $D_o(\hat{I},E)$.

If $\phi = y\oplus\phi'$ a.e. on E, then $f(\phi) = f(y) + f(\phi')$ a.e. on E for each f in Y'. Hence, ϕ is weakly measurable and $I\left(f(\phi),E\right) = I\left(f(y),E\right) + I\left(f(\phi'),E\right) = f(y) + 0$ for every $f\epsilon Y'$. Therefore $\phi\epsilon D_o(\hat{I},E)$.

(2) Let $\phi\epsilon D_o(\hat{I},E)$ and $\hat{I}(\phi,E) = y\oplus B$. If $\phi_y(x) = y$ for every $x\epsilon X$, then $\hat{I}(\phi_y,E) = \hat{I}(\phi,E)$. We now show that this implies that ϕ and ϕ_y must belong to the same coset of K(E) in $D_o(\hat{I},E)$.

Theorems 5 and 16 imply that $\hat{I}(\cdot,E)$ is a homomorphism of $D_o(\hat{I},E)$ onto Y/B. The kernel of the homomorphism is K(E). Therefore $\hat{I}(\cdot,E)$ is an isomorphism of $D_o(\hat{I},E)/K(E)$ onto Y/B. Hence ϕ and ϕ_y belong to the same coset of K(E) because $\hat{I}(\phi,E) = \hat{I}(\phi_y,E)$.

Assume that $\phi,\phi_y\epsilon\bar{\phi}\oplus K(E)$ for some $\bar{\phi}\epsilon D_o(\hat{I},E)$. Then $\phi\epsilon\phi_y\oplus K(E)$. Therefore there is a $\phi'\epsilon K(E)$ so that $\phi = \phi_y\oplus\phi'$. Since $K_o(E) \subset K(E)$, ϕ' may belong to $K_o(E)$. □

The following theorem and its corollary extend the results of Theorem 7 and its corollary.

Theorem 19

(a) Assume that Y' is countable and $\phi_1,\phi_2 \epsilon D(\hat{I})$. If $\hat{I}(\phi_1,E) = \hat{I}(\phi_2,E)$ for every $E\epsilon A$, then there is a $\phi'\epsilon K_o(X)$ so that $\phi_1 = \phi'\oplus\phi_2$.

(b) If $\phi_1,\phi_2 \epsilon D_o(\hat{I},E)$ and $\hat{I}(\phi_1,E) = \hat{I}(\phi_2,E)$, then $\phi_1 = \phi'\oplus\phi_2$ for some $\phi'\epsilon K(E)$.

Proof

(a) For each $f\epsilon Y'$ we have $I\left(f(\phi_1-\phi_2),E\right) = 0$ for every $E\epsilon A$. Hence $f(\phi_1-\phi_2) = 0$ a.e. on X for every $f\epsilon Y'$. Theorem 7 implies that $\phi_1 - \phi_2 \epsilon B$ a.e. on X. If $\phi' = \phi_1-\phi_2$, then $\phi'\epsilon K_o(X)$ and $\phi_1 = \phi'\oplus\phi_2$.

(b) Let $y\epsilon\hat{I}(\phi_1,E)$. Then $\hat{I}(\phi_1,E) = \hat{I}(\phi_2,E) = y\oplus B$. Hence $\hat{I}(\phi_1-\phi_2,E) = B$. Therefore $\phi_1 - \phi_2\epsilon K(E)$. If $\phi' = \phi_1-\phi_2$, then $\phi'\epsilon K(E)$ and $\phi_1 = \phi'\oplus\phi_2$. □

Corollary 1 to Theorem 19. Assume that Y' is closed under addition and multiplication by real numbers. Let H be a Hamel basis for Y'. If H is countable and $\hat{I}(\phi_1,E) = \hat{I}(\phi_2,E)$ for every $E\epsilon A$, then $\phi_1 = \phi'\oplus\phi_2$ for some ϕ' in $K_o(X)$.

Proof

We have $f(\phi_1-\phi_2) = 0$ a.e. on X for every $f\epsilon Y'$. Corollary 1 to Theorem 7 implies that $\phi_1 - \phi_2\epsilon B$ a.e. on X. If $\phi' = \phi_1-\phi_2$, then $\phi'\epsilon K_o(X)$ and $\phi_1 = \phi'\oplus\phi_2$. □

Remark on Theorem 19. Theorem 19 and its corollary generalize Theorem 5.2 of Pettis ([2], p.293). Pettis' theorem states: if ϕ_1,ϕ_2 are strongly measurable and Pettis integrable, then $P(\phi_1,E) = P(\phi_2,E)$ for every $E\epsilon A$ if and only if $\phi_1 = \phi_2$ a.e. on X.

We will now consider the case where the group Y is also partially ordered.

Y a Partially Ordered Group

In this section Y will be a group which is also a partially ordered set, with or without a topology.

We will discuss convergence properties, absolute continuity, and other properties of $\hat{I}$. Absolute value will be defined for elements in Y and this will enable us to improve the dominated convergence theorem (Theorems 1 and 8), the absolute continuity theorem (Theorem 4), and to obtain other order properties of $\hat{I}$.

Definition 15

(a) A p.o. set Y is a lattice (Y is a ℓ.o. set) if and only if for any $y_1, y_2 \in Y$ the least upper bound $(y_1 \vee y_2)$ and the greatest lower bound $(y_1 \wedge y_2)$ are uniquely determined elements in Y.

(b) Y is a p.o. (f.o., ℓ.o.) group if and only if: (1) Y is a group; and (2) Y is a p.o. (f.o., ℓ.o.) set.

(c) Y is a p.o. (f.o., ℓ.o.) group topological space if and only if: (1) Y is a p.o. (f.o., ℓ.o.) group; and (2) Y is a topological space.

(d) If Y is a p.o. group topological space, then COH(Y,R) is the set of all continuous order preserving homomorphisms of Y into R.

Our definition of a p.o. group is not standard. Fuchs ([25], p.9) defines a p.o. group by adding the following third condition: if $y_1 \leq y_2$, then $y_1 \oplus y \leq y_2 \oplus y$ and $y \oplus y_1 \leq y \oplus y_2$ for every $y \in Y$. We do not assume there is any relation between $\leq$ and $\oplus$ in a p.o. group except in Theorem 20.

The set COH(Y,R) is non-empty because the function which is identically zero on Y belongs to COH(Y,R). We will now define absolute value on Y.

Definition 16

(a) If Y is a f.o. group, then $|y| = \max(y,-y)$ for every $y \in Y$.

(b) If Y is a ℓ.o. group, then $|y| = y \vee (-y)$ for every $y \in Y$.

(c) If Y is a f.o. (ℓ.o.) group and $\phi \in (X,Y)$, then $|\phi|(x) = |\phi(x)|$ for every $x \in X$.

If Y is a p.o. (ℓ.o.) group, then so is (X,Y). But if Y is a f.o. group, then (X,Y) is a p.o. group. It is easy to see that the two definitions of $|y|$ agree whenever Y is both a ℓ.o. group and a f.o. group.

The following lemmas will be used later on in this section.

Lemma 1. If Y is a ℓ.o. group and $f \in OH(Y,R)$, then $|f(y)| \leq f(|y|)$ for every $y \in Y$.

Proof. Since $-y \leq |y|$ and $y \leq |y|$, we have $-f(y) = f(-y) \leq f(|y|)$ and $f(y) \leq f(|y|)$. Therefore $|f(y)| = f(y) \vee \left(-f(y)\right) \leq f(|y|)$ for every $y \in Y$. $\square$

Lemma 2. If Y is a f.o. group and $f \in OH(Y,R)$, then $|f(y)| = f(|y|)$ for every $y \in Y$.

Proof. Since $-y \leq |y|$ and $y \leq |y|$, we have $|f(y)| \leq f(|y|)$ for every $y \in Y$. But $|y| = y$ or $-y$. If $|y| = y$, then $f(|y|) = f(y) \leq |f(y)|$ and if $|y| = -y$, then $f(|y|) = -f(y) \leq |f(y)|$. In either case $f(|y|) \leq |f(y)|$ for every $y \in Y$. Hence $|f(y)| = f(|y|)$ for every $y \in Y$. □

Lemma 3. If Y is a f.o. (ℓ.o.) group, $\phi_1, \phi_2 \in (X,Y)$ and $|\phi_1| \leq \phi_2$ a.e. on X, then $|f(\phi_1)| \leq f(\phi_2)$ a.e. on X for every $f \in OH(Y,R)$.

Proof. Let $f \in OH(Y,R)$. Then, from Lemmas 1 and 2, we have $|f(\phi_1(x))| \leq f(|\phi_1(x)|) = f(|\phi_1|(x)) \leq f(\phi_2(x))$ a.e. on X. Hence $|f(\phi_1)| \leq f(\phi_2)$ a.e. on X. □

Lemma 4. Let Y be a f.o. group and let $y_1, y_2 \in Y$ with $f(y_1 - y_2) \geq 0$ for every $f \in Y' \subset OH(Y,R)$. Then $y_2 \oplus B \leq y_1 \oplus B$.

Proof. Recall (Definition 10) that $y_2 \oplus B \leq y_1 \oplus B$ if and only if $y_2 \oplus B = y_1 \oplus B$ or $y_2' < y_1'$ for every $y_2' \in y_2 \oplus B$ and for every $y_1' \in y_1 \oplus B$.

It suffices to show the following. If there is a $y_2' \in y_2 \oplus B$ and a $y_1' \in y_1 \oplus B$ such that $y_1' \leq y_2'$, then $y_1 \oplus B = y_2 \oplus B$.

Assume that $y_1' \leq y_2'$. Then $f(y_1') \leq f(y_2')$ for every $f \in Y'$. Therefore $f(y_2' - y_1') \geq 0$ for every $f \in Y'$. Hence $f(y_1' - y_2') \leq 0$ for every $f \in Y'$. It follows that $f(y_1' - y_2') = 0$ for every $f \in Y'$, because $f(y_1' - y_2') = f(y_1 - y_2) \geq 0$ for every $f \in Y'$. This implies that $y_1' - y_2' \in B$, from which it follows that $y_1 \oplus B = y_2 \oplus B$. □

Convergence Theorems

We now extend Definition 13 to series whose terms are functions. Let h be a function on the set of positive integers into {0,1} and let

$\phi_i \epsilon (X,Y)$ for $i=1,2,\cdots$. Define $s_1 = \phi_1$ and define s_n, $n\geq 2$, inductively as follows: $s_n = s_{n-1}\oplus\phi_n$ if $h(n) = 1$ and $s_n = \phi_n \oplus s_{n-1}$ if $h(n) = 0$. We will write $h\sum_{i=1}^{n} \phi_i$ for s_n.

In the next theorem and its corollary we will assume that Y is a p.o. group topological space with the following additional property: if $y_1 \leq y_2$, then $y_1\oplus y \leq y_2\oplus y$ and $y\oplus y_1 \leq y\oplus y_2$ for every y in Y.

Theorem 20. Assume that Y is a p.o. group topological space, $Y' \subset COH(Y,R)$, and that for each $i=1,2,\cdots$, $\phi_i \epsilon (X,Y)$ is weakly measurable with $\theta \leq \phi_i$ a.e. on X. If $\phi \epsilon D(\hat{I})$ and $h\sum_{i=1}^{n} \phi_i \to \phi$ a.e. on X, then $\sum_{i=1}^{n} \hat{I}(\phi_i) \uparrow \hat{I}(\phi)$.

Proof

(a) We first show that $\theta \leq s_1 \leq s_2 \leq \cdots \to \phi$ a.e. on X. Since $s_1 = \phi_1$, it follows that $\theta \leq s_1$ a.e. on X. Assume that $n\geq 2$ and $h(n) = 1$. Since $\theta \leq \phi_n$ a.e. on X, we have $s_{n-1} \leq s_{n-1}\oplus\phi_n = s_n$ a.e. on X. If $h(n) = 0$, then $s_{n-1} \leq \phi_n \oplus s_{n-1} = s_n$ a.e. on X.

(b) It follows that $0 \leq f(s_1) \leq f(s_2) \leq \cdots \to f(\phi)$ a.e. on X for each f in Y'. Hence $0 \leq f(s_n) \leq f(\phi)$ a.e. on X for each n and each f in Y'. Therefore $s_n \epsilon D(\hat{I})$ for all n. But $f(\phi_n) = f(s_n) - f(s_{n-1})$ for $n\geq 2$ and each f in Y'. This implies that $\phi_n \epsilon D(\hat{I})$ for all n.

(c) The monotone convergence theorem (Theorem 10) applied to the sequence $s_1, s_2, s_3, \cdots$ says that $\hat{I}(s_1) \leq \hat{I}(s_2) \leq \cdots \to \hat{I}(\phi)$. But $\hat{I}(s_n) = \sum_{i=1}^{n} \hat{I}(\phi_i)$ by Theorem 13. □

Corollary 1 to Theorem 20. Assume now that the ϕ_i are in $D_o(\hat{I})$ and that Y is weakly sequentially complete. Then $\phi \epsilon D_o(\hat{I})$ and if $y_i \epsilon \hat{I}(\phi_i)$

for $i=1,2,\cdots$, and $y \in \hat{I}(\phi)$, then $k \sum_{i=1}^{n} y_i \xrightarrow{w} y$ for every k mapping the positive integers into $\{0,1\}$.

Proof. Let k be any function mapping the positive integers into $\{0,1\}$ and let $F = \hat{I}(\phi)$. Theorem 20 implies that $\lim_n \sum_{i=1}^{n} F_{y_i}(f) = \lim_n \sum_{i=1}^{n} f(y_i) = \lim_n f\left(k \sum_{i=1}^{n} y_i\right) = F(f)$ for each f in Y'. Therefore there is a $\bar{y} \in Y$ so that $f\left(k \sum_{i=1}^{n} y_i\right) \to f(\bar{y})$ for every $f \in Y'$. Hence $\bar{y} \in \hat{I}(\phi)$ and $\phi \in D_o(\hat{I})$. If $y \in \hat{I}(\phi)$, then $f(y) = f(\bar{y})$ for each f in Y'. Therefore $k \sum_{i=1}^{n} y_i \xrightarrow{w} y$ also. $\square$

Theorem 21. (Dominated Convergence Theorem.) Assume that Y is a f.o. or ℓ.o. group topological space, $Y' \subset COH(Y,R)$, and that for each $i=1,2,3,\cdots$, $\phi_i \in (X,Y)$ is weakly measurable and $|\phi_i| \leq \bar{\phi}$ a.e. on X with $\bar{\phi} \in D(\hat{I})$. If $\phi_i \to \phi$ a.e. on X or $f(\phi_i)$ converges to $f(\phi)$ in measure on X for every $f \in Y'$, then $\phi \in D(\hat{I})$ and $\hat{I}(\phi_n) \to \hat{I}(\phi)$.

Proof. Lemma 3 says that $|f(\phi_i)| \leq f(\bar{\phi})$ a.e. on X for all i and for all $f \in Y'$. Hence each $\phi_i \in D(\hat{I})$. Let $F_i = \hat{I}(\phi_i)$ for $i=1,2,3,\cdots$.

The dominated convergence theorem for the Lebesgue integral implies that $F_i(f) = I\{f(\phi_i)\} \to I\{f(\phi)\}$ and $|I\{f(\phi)\}| < +\infty$ for each $f \in Y'$. Hence $\phi \in D(\hat{I})$. If $F = \hat{I}(\phi)$, then $F(f) = I\{f(\phi)\}$ for all $f \in Y'$. Therefore $F_i \to F$ (pointwise). $\square$

Corollary 1 to Theorem 21. Assume the same hypothesis as in Theorem 21 except now each $\phi_i \in D_o(\hat{I})$ and Y is weakly sequentially complete. Then $\phi \in D_o(\hat{I})$ and if $y_i \in \hat{I}(\phi_i)$ for $i=1,2,3,\cdots$ and $y \in \hat{I}(\phi)$, then $y_i \xrightarrow{w} y$.

Proof. Let $y_i \epsilon \hat{I}(\phi_i)$ for $i=1,2,3,\cdots$. Theorem 21 implies that $f(y_i) \to I\big(f(\phi)\big)$ (finite) for each $f\epsilon Y'$. Therefore there is a $\bar{y}\epsilon Y$ such that $y_i \xrightarrow{w} \bar{y}$. Hence $\bar{y}\epsilon\hat{I}(\phi)$ and $\phi\epsilon D_o(\hat{I})$. □

Corollary 2 to Theorem 21. Assume now that Y is a f.o. group topological space. Then $\hat{I}(|\phi_i-\phi|) \to 0$, where $0(f) = 0$ for all $f\epsilon Y'$.

Proof. Lemma 2 implies that $|f(\phi_i) - f(\phi)| = |f(\phi_i-\phi)| = f(|\phi_i-\phi|)$ for all i and f in Y'. Therefore each $|\phi_i-\phi|$ is in $D(\hat{I})$. Let $F_i = \hat{I}(|\phi_i-\phi|)$ for $i=1,2,\cdots$.

The dominated convergence theorem for the Lebesgue integral ([23], p.125) implies that $F_i(f) = I\big(f(|\phi_i-\phi|)\big) = I\big(|f(\phi_i)-f(\phi)|\big) \to 0$ for each $f\epsilon Y'$. Hence $F_i \to 0$ (pointwise). □

Theorem 22. (Monotone Convergence Theorem.) Assume that Y is a f.o. group topological space, $Y' \subset COH(Y,R)$, $\phi_i\epsilon D_o(\hat{I})$ for $i=1,2,3,\cdots$, $\phi\epsilon D(\hat{I})$, $\phi_1\le\phi_2\le \cdots \to \phi$, and that Y is weakly sequentially complete. Then $\phi\epsilon D_o(\hat{I})$ and if $y_i\epsilon\hat{I}(\phi_i)$ for $i=1,2,3,\cdots$ and $y\epsilon\hat{I}(\phi)$, then $y_1\oplus B \le y_2\oplus B \le \cdots \xrightarrow{w} y\oplus B$.

Proof. If $A_1,A_2,\cdots$ are subsets of Y, then $A_i \xrightarrow{w} A$ a subset of Y if and only if for any y_i in A_i and any y in A we have $y_i \xrightarrow{w} y$.

Let $y_i\epsilon\hat{I}(\phi_i)$ for $i=1,2,\cdots$. Theorem 10 implies that $f(y_1) \le f(y_2) \le \cdots \to I\big(f(\phi)\big)$ for every $f\epsilon Y'$. Therefore there is a $\bar{y}\epsilon Y$ so that $f(y_i) \to f(\bar{y})$ for each f in Y'. Hence $\bar{y}\epsilon\hat{I}(\phi)$ and $\phi\epsilon D_o(\hat{I})$.

If f is in Y', then $f(y_{n+1}-y_n) \ge 0$ for all $n\ge1$ because $f(y_n) \le f(y_{n+1})$ for all $n\ge1$. Lemma 4 implies that $y_n\oplus B \le y_{n+1}\oplus B$ for $n\ge1$. □

Absolute Continuity of $\hat{I}$

Theorem 23. Assume that Y is a f.o. group, $Y' \subset OH(Y,R)$, $E_i \epsilon A$ for $i=1,2,\cdots$, and that $\mu(E_i) \to 0$. If $\phi \epsilon D(\hat{I})$, then $\hat{I}(|\phi|,E_i) \to 0$, where $0(f) = 0$ for all f in Y'.

Proof. Lemma 2 implies that $|f(\phi)| = f(|\phi|)$ for all f in Y'. Therefore $|\phi| \epsilon D(\hat{I})$. Let $F_i = \hat{I}(|\phi|,E_i)$ for $i=1,2,\cdots$.

The absolute continuity of the Lebesgue integral implies that ([23], p.124) $F_i(f) = I\big(f(|\phi|),E_i\big) = I\big(|f(\phi)|,E_i\big) \to 0$ for each $f \epsilon Y'$. Therefore $F_i \to 0$ (pointwise). □

Corollary 1 to Theorem 23. Assume now that $\phi \epsilon D_o(\hat{I},E_i)$ for $i=1,2,3,\cdots$. If $\phi \epsilon D(\hat{I})$ and $y_i \epsilon \hat{I}(\phi,E_i)$ for $i=1,2,3,\cdots$, then $|y_i| \xrightarrow{w}$ each $y \epsilon B$.

Proof. For each $f \epsilon Y'$ we have $f(|y_i|) = |f(y_i)| = |I\big(f(\phi),E_i\big)| \leq I\big(|f(\phi)|,E_i\big) \to 0$. If $y \epsilon B$, then $f(|y_i|) \to f(y) = 0$ for every $f \epsilon Y'$. Hence $|y_i| \xrightarrow{w}$ each $y \epsilon B$. □

Other Properties of $\hat{I}$

Here we will use the absolute value defined on (X,Y) to deduce order properties of $\hat{I}$.

Theorem 24. Assume that $Y' \subset OH(Y,R)$ and that $\phi \epsilon (X,Y)$ is weakly measurable.

(a) If Y is a f.o. group, then $\phi \epsilon D(\hat{I})$ if and only if $|\phi| \epsilon D(\hat{I})$.

(b) If Y is a ℓ.o. group, then $|\phi| \epsilon D(\hat{I})$ implies that $\phi \epsilon D(\hat{I})$.

Proof

(a) Lemma 2 says that $|f(\phi)| = f(|\phi|)$ for every $f \epsilon Y'$.

(b) Lemma 1 says that $|f(\phi)| \le f(|\phi|)$ for every $f \in Y'$. □

Theorem 25. Let $Y' \subset OH(Y,R)$ and let $\phi \in (X,Y)$ be weakly measurable. Assume either (a) Y is a f.o. group and $\phi \in D(\hat{I})$, or (b) Y is a ℓ.o. group and $|\phi| \in D(\hat{I})$. Then $|\hat{I}(\phi)| \le \hat{I}(|\phi|)$.

Proof. In either case Theorem 24 implies that ϕ and $|\phi|$ belong to $D(\hat{I})$. Let $F_1 = \hat{I}(\phi)$ and $F_2 = \hat{I}(|\phi|)$.

Theorem 11 implies that $F_1 \le F_2$ because $\phi \le |\phi|$. Also $-\phi \le |\phi|$ implies that $-F_1 = -\hat{I}(\phi) = \hat{I}(-\phi) \le F_2$. Hence $-F_2 \le F_1 \le F_2$. Therefore $|F_1| \le F_2$. □

Corollary 1 to Theorem 25. Let Y be a f.o. group, $Y' \subset OH(Y,R)$, ϕ and $|\phi|$ belong to $D_o(\hat{I})$, and let $y_1 \in \hat{I}(\phi)$ and $y_2 \in \hat{I}(|\phi|)$. Then $|\hat{I}(\phi)| = |y_1 \oplus B| \subset (y_1 \oplus B) \cup (-y_1 \oplus B)$ and $y_1 \oplus B \le y_2 \oplus B$, $-y_1 \oplus B \le y_2 \oplus B$.

Proof

(a) Let $B_1 = \{b \in B \mid -(y_1 \oplus b) \le y_1 \oplus b\}$ and $B_2 = \{b \in B \mid y_1 \oplus B < -(y_1 \oplus b)\}$. Then $|y_1 \oplus B| = \{y_1 \oplus b \mid b \in B_1\} \cup \{-y_1 - b \mid b \in B_2\} \subset (y_1 \oplus B) \cup (-y_1 \oplus B)$.

(b) Theorem 25 implies that $f(y_1) \le |f(y_1)| \le f(y_2)$ for each $f \in Y'$. Hence $f(y_2 - y_1) \ge 0$ for each $f \in Y'$. Lemma 4 implies that $y_1 \oplus B \le y_2 \oplus B$.

(c) Theorem 25 implies that $f(-y_1) = -f(y_1) \le |f(y_1)| \le f(y_2)$ for each $f \in Y'$. Hence $f\big(y_2 - (-y_1)\big) \ge 0$ for each $f \in Y'$. Lemma 4 implies that $-y_1 \oplus B \le y_2 \oplus B$. □

<u>Theorem 26</u>. Assume that Y is a f.o. or ℓ.o. group, $Y' \subset OH(Y,R)$, $\phi_1 \epsilon (X,Y)$ is weakly measurable, and that $\phi_2 \epsilon D(\hat{I})$. If $|\phi_1| \le \phi_2$ a.e. on X, then $\phi_1 \epsilon D(\hat{I})$.

<u>Proof</u>. Lemma 3 implies that $|f(\phi_1)| \le f(\phi_2)$ a.e. on X for each $f \epsilon Y'$. Hence $\phi_1 \epsilon D(\hat{I})$. □

<u>Theorem 27</u>. Assume that Y is a f.o. group, $Y' \subset OH(Y,R)$, $\phi \epsilon D_o(\hat{I})$, and that $\phi \ge \theta$ a.e. on X. Then $\hat{I}(\phi) \ge B$.

<u>Proof</u>. Let $y \epsilon \hat{I}(\phi)$. Then $f(y) = I\big(f(\phi)\big) \ge 0$ for each $f \epsilon Y'$. Let $y' \epsilon B$. Then $f(y-y') = f(y) - f(y') = f(y) \ge 0$ for each $f \epsilon Y'$. Lemma 4 implies that $\hat{I}(\phi) = y \oplus B \ge y' \oplus B = B$. □

<u>Theorem 28</u>. Assume that Y' is a p.o. group, $Y' \subset O(Y,R)$, Y' is countable, $\phi \epsilon D(\hat{I},E)$, $\mu(E) > 0$, and that $\phi \ge \theta$ a.e. on E. If $\hat{I}(\phi,E) = 0 \epsilon (Y',R)$, then $\phi \epsilon B$ a.e. on E.

<u>Proof</u>. For each f in Y' we have $f(\phi) \ge 0$ a.e. on E. Since $I\big(f(\phi),E\big) = 0$ for each f in Y', we see that $f(\phi) = 0$ a.e. on E for every f in Y'. Let $Y' = \{f_1, f_2, \cdots\}$.

Let $A_i = \overset{-1}{\phi}\big(\overset{-1}{f_i}(0)\big)$ for $i=1,2,\cdots$. Then $\bigcap_{i=1}^{\infty} A_i = \overset{-1}{\phi}(B)$. Now $\mu\big(E - \bigcap_{i=1}^{\infty} A_i\big) = 0$ because $\mu(E-A_i) = 0$ for all i. If B is empty, then $\mu(E) = 0$. Hence B is non-empty and $\phi \epsilon B$ a.e. on E. □

CHAPTER V

THE COMPLEX NUMBERS UNDER MULTIPLICATION AS THE RANGE SPACE OF THE FUNCTIONALS

In this chapter (except in part (c) of Theorem 5) the range space of the functions to be integrated, Y, will be a locally compact, Hausdorff, Abelian topological group. We will write $\oplus$ and $-$ for addition and subtraction in Y, θ for the identity of Y, and we always assume that Y has at least two elements. The complex numbers under multiplication (denoted by $C_{\bullet}$) will always be the range space Z of the functionals. The reason for choosing $C_{\bullet}$ for the range space of the functionals is that then Pontrjagin's duality theorem (Theorem 1) says that Y is the same as its "double dual."

In Lemma 1 we note that each functional maps Y into the set T of complex numbers with absolute value one. That is, each functional maps Y into the locally compact, Hausdorff, Abelian topological group T. Let H be any locally compact, Hausdorff, Abelian topological group and assume that H serves as the range space of the functionals. Then the duality theorem (Theorem 1) holds if and only if $H = T$ ([21], p.424). In this sense $C_{\bullet}$ is the only space, which contains the set of complex numbers, that we could use for the range space of the functionals so that the duality theorem holds. The range space of the functionals must contain the set of complex numbers because in this chapter the set of complex numbers will be the range space of the Lebesgue integral.

We will employ the same notation as in Chapter IV. X is a non-empty set, $\mathcal{A}$ a σ-algebra of subsets of X, and μ is a complex measure (see Appendix B) on $\mathcal{A}$. We always assume that μ is <u>not identically zero</u>. $I(\psi,A)$ denotes the Lebesgue integral of measurable complex-valued ψ over $A \in \mathcal{A}$ with respect to complex measure μ. The domain of $I(\cdot,A)$ is $D(I,A) = \{\psi \mid |I(\psi,A)| < +\infty\}$. We write $D(I)$ for $D(I,X)$. Properties of this integral and its definition are presented in Appendix B.

In the first section we define and discuss the type of functional we will be using in this chapter. The duality theorem is stated in this section. In the second section we show that when $C_{\bullet}$ is the range space of the functionals we get a useful* extension of the Lebesgue integral if and only if μ is a zero-one measure.

In Chapter IV we saw that $\hat{I}$ inherited most of the basic properties of I, but we were able to say very little about when a function ϕ belonged to $D(\hat{I})$ or to $D_o(\hat{I})$. In this chapter we will show that we can say much more about when a function ϕ belongs to $D(\hat{I})$ or $D_o(\hat{I})$, but only when μ is a zero-one measure will $\hat{I}$ be a "useful" extension of I.

Pontrjagin's Duality Theorem

In this section we will discuss the functionals we will be using in this chapter and we will state Pontrjagin's duality theorem.

<u>Definition 1</u>

(a) $C_{\bullet}$ is the set of complex numbers under multiplication.

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*The sense in which the extension is useful is indicated by Theorems 2, 3, 4, and 5.

(b) C_+ is the set of complex numbers under addition.

(c) $C_\bullet^+$ is $C_\bullet$ without zero.

(d) T is the subgroup of $C_\bullet^+$ consisting of all the complex numbers with absolute value one.

Note that C_+, $C_\bullet^+$, and T are all locally compact, Hausdorff, Abelian topological groups but $C_\bullet$ is not even a group. Throughout this chapter, except in part (c) of Theorem 5, Y' is always the set of all bounded, not identically zero, continuous homomorphisms of Y into $C_\bullet$. Y' separates points in Y ([21], p.345).

Mackey [26] has considered functionals which are not bounded. Let Y_1' be the set of all not identically zero continuous homomorphisms of Y into $C_\bullet$ and let Y_2' be the set of all positive continuous homomorphisms of Y into the reals under multiplication. The elements in Y' are called characters of Y, members of Y_1 are called generalized characters of Y, and functions in Y_2' are real characters of Y. Mackey ([26], p.157) noted that g is a generalized character of Y if and only if there is a character f and a real character h so that g = (f)(h).

We will write $\mathit{1}$ for the function in Y' which maps Y onto 1. If f_1 and f_2 are in Y', then we define $(f_1f_2)(y) = f_1(y)f_2(y)$ for every y in Y. Similarly we define multiplication in $(Y',C_\bullet)$.

Lemma 1. Each f in Y' maps Y into T.

Proof. Let f be in Y'.

(a) Assume that there is a y in Y such that $|f(y)| > 1$. Let y_n be the sum of y with itself n times. Then $|f(y_n)| = |f(y)|^n \to +\infty$ and f is not bounded.

(b) Surely $f(\theta) = 1$. Assume that there is a $y \in Y$ so that $|f(y)| < 1$. Then $1 = |f(y \oplus (-y))| = |f(y)||f(-y)|$ and $|f(-y)| > 1$. □

If $f \in Y'$, then let $\bar{f}(y) = \overline{f(y)}$ for every $y \in Y$. Lemma 1 implies that $f^{-1} = \bar{f}$ and $f(-y) = \overline{f(y)}$ for every $y \in Y$, for each $f \in Y'$.

Y', with the compact open topology ([20], p.221), is a locally compact, Hausdorff, Abelian topological group ([21], p.361). Let Y'' be the set of all bounded, not identically zero, continuous homomorphisms of Y' into $C_{\bullet}$. Lemma 1 implies that each F in Y'' maps Y' into T. Y'', with the compact open topology, is a locally compact, Hausdorff, Abelian topological group. Recall that $\lambda(y) = F_y$ for each $y \in Y$, where $F_y(f) = f(y)$ for each $f \in Y'$.

Theorem 1. (Pontrjagin's Duality Theorem.) The canonical map λ is a homeomorphism and an isomorphism of Y onto Y''.

Proof. For a proof of this theorem see Hewitt and Ross ([21], p.378). □

In the rest of this chapter Y' and Y'' will have the compact open topology.

Zero-One Measures

By a zero-one measure we mean a measure μ on a σ-algebra A of subsets of X whose only values are 0 and 1 with $\mu(X) = 1$. In this section we show that a useful generalization of the Lebesgue integral, when $C_{\bullet}$ is used as the range space of the functionals, is obtained only when μ is a zero-one measure.

Theorem 2. $\hat{I}(\phi_1 \oplus \phi_2) = \hat{I}(\phi_1)\hat{I}(\phi_2)$ on $D(\hat{I})$ if and only if μ is a zero-one measure.

Remark on Theorem 2. Theorem 2 says that the extension $\hat{I}(\cdot)$ of the Lebesgue integral is additive if and only if μ is a zero-one measure. Recall that $\hat{I}(\cdot)$ maps $D(\hat{I})$ into $(Y', C_\bullet)$ and $\hat{I}(\phi_1)\hat{I}(\phi_2)$ is multiplication in $(Y', C_\bullet)$.

Proof

(1) Assume that $\hat{I}(\phi_1 \oplus \phi_2) = \hat{I}(\phi_1)\hat{I}(\phi_2)$ on $D(\hat{I})$. Since $\mathit{1} \in Y'$ we see that $\mu(X) = I\{\mathit{1}(\phi_1 \oplus \phi_2)\} = \hat{I}(\phi_1 \oplus \phi_2)(\mathit{1}) = \{\hat{I}(\phi_1)(\mathit{1})\}\{\hat{I}(\phi_2)(\mathit{1})\} = I\{\mathit{1}(\phi_1)\}I\{\mathit{1}(\phi_2)\} = \mu(X)\mu(X)$. Hence $\mu(X)$ is either zero or one. We will show that $\mu(X)$ can't be zero, but first we take up the case where $\mu(X) = 1$.

(a) Assume that $\mu(X) = 1$. We will now show that $\mu(A)$ is zero or one for every $A \in \mathcal{A}$.

Let $A \in \mathcal{A}$. Assume that y_1 and y_3 are arbitrary in Y and let $y_2 = -y_3$ and $y_4 = -y_1$. Let $\phi_1, \phi_2 \in (X,Y)$ where $\phi_1 = y_1$ on A and y_2 on A^c, and $\phi_2 = y_3$ on A and y_4 on A^c. Surely $\phi_1, \phi_2 \in D(\hat{I})$ because $\mu(X) = 1$.

Now for each $f \in Y'$ we have $I\{f(\phi_1 \oplus \phi_2)\} = f(y_1 \oplus y_3)\mu(A) + f(y_2 \oplus y_4)\mu(A^c)$. Also for each $f \in Y'$ we see that $I\{f(\phi_1)\}I\{f(\phi_2)\} = f(y_1 \oplus y_3)\{\mu(A)\}^2 + f(y_2 \oplus y_4)\{\mu(A^c)\}^2 + \mu(A)\mu(A^c)\{f(y_1 \oplus y_4) + f(y_2 \oplus y_3)\}$ because f is a homomorphism. Since $\hat{I}(\phi_1 \oplus \phi_2)(f) = I\{f(\phi_1 \oplus \phi_2)\} = I\{f(\phi_1)\}I\{f(\phi_2)\} = \{\hat{I}(\phi_1)\hat{I}(\phi_2)\}(f)$ holds for each $f \in Y'$, we have:

$$f(y_1 \oplus y_3)\mu(A) + f(y_2 \oplus y_4)\mu(A^c) = f(y_1 \oplus y_3)\big(\mu(A)\big)^2 +$$

$$f(y_2 \oplus y_4)\big(\mu(A^c)\big)^2 + \mu(A)\mu(A^c)\big(f(y_1 \oplus y_4) + f(y_2 \oplus y_3)\big) \qquad (1)$$

holding for each $f \epsilon Y'$. Let $\Gamma = f(y_1 \oplus y_3) + f(y_2 \oplus y_4) - f(y_1 \oplus y_4) - f(y_2 \oplus y_3)$. Equation (1) becomes $\mu(A)\big(1-\mu(A)\big)\Gamma = 0$ for every $f \epsilon Y'$ because $\mu(A^c) = 1 - \mu(A)$.

We now show that Γ can't be zero for all $f \epsilon Y'$, implying that $\mu(A)$ is zero or one. Assume that $\Gamma = 0$ for all $f \epsilon Y'$. Let $f \epsilon Y'$ and $f \neq \mathit{1}$. Such an f exists because Y' separates points in Y. Therefore $0 = \Gamma = f(y_1 \oplus y_3) + f\big(-(y_1 \oplus y_3)\big) - f(\theta) - f(\theta) = f(y_1 \oplus y_3) + \overline{f(y_1 \oplus y_3)} - 2$ because $f(-y) = \overline{f(y)}$. Let us write $a + bi$ for $f(y_1 \oplus y_3)$. Hence $(a+bi) + (a-bi) = 2$, or $a=1$. Since $|f(y)| = 1$ for every $y \epsilon Y$ (Lemma 1), we see that $b=0$. This implies that $f=\mathit{1}$ because y_1 and y_3 were arbitrary in Y. Therefore $\mu(A)$ must be zero or one.

(b) Now assume that $\mu(X) = 0$. We will show that this implies $\mu(A) = 0$ for every $A \epsilon \mathcal{A}$. This is a contradiction because we have assumed that μ is not identically zero.

Let $A \epsilon \mathcal{A}$. Assume that y_1 and y_2 are arbitrary in Y and let $y_3 = -y_1$ and $y_4 = -y_2$. Define ϕ_1 and ϕ_2 as in part (a) above. Equation (1) now becomes

$$\mu(A)\big(f(y_1 \oplus y_3) - f(y_2 \oplus y_4) - \mu(A)\Gamma\big) = 0 \qquad (2)$$

for every $f \epsilon Y'$ because $\mu(A^c) = -\mu(A)$.

Now assume that $\mu(A) \neq 0$ and we will arrive at a contradiction. If $\mu(A) \neq 0$, then Equation (2) becomes $f(y_1 \oplus y_3) - f(y_2 \oplus y_4) = \mu(A)\Gamma$ for every $f \epsilon Y'$. Let $f \epsilon Y'$ and $f \neq 1$. Therefore $0 = f(\theta) - f(\theta) = f(y_1 - y_1) - f(y_2 - y_2) = \mu(A)\left[f(\theta) + f(\theta) - f(y_1 - y_2) - f\left(-(y_1 - y_2)\right)\right]$. Hence $\overline{f(y_1 - y_2)} + f(y_1 - y_2) = 2$ because $f(-y) = \overline{f(y)}$ and $\mu(A) \neq 0$. If $f(y_1 - y_2) = a + bi$, then $(a+bi) + (a-bi) = 2$. Hence $a=1$. It follows that $b=0$ because $|f(y)| = 1$ (Lemma 1) for every $y \epsilon Y$. Therefore $f=1$ because y_1 and y_2 were arbitrary in Y.

(2) Now assume that μ is a zero-one measure. Let $\phi_1, \phi_2 \epsilon D(\hat{I})$. We wish to show that $\hat{I}(\phi_1 \oplus \phi_2) = \hat{I}(\phi_1)\hat{I}(\phi_2)$. This last equation holds if and only if

$$I\left[f(\phi_1)f(\phi_2)\right] = I\left[f(\phi_1 \oplus \phi_2)\right] = I\left[f(\phi_1)\right] I\left[f(\phi_2)\right] \tag{3}$$

for every f in Y'. We will denote the real part of $f(\phi_i)$ by $\left[f(\phi_i)\right]_1$ and the imaginary part by $\left[f(\phi_i)\right]_2$ for each f in Y' and for $i=1,2$. Let $L_{ij} = \left[f(\phi_i)\right]_j$ for $i=1,2$ and $j=1,2$. Comparing the real and imaginary parts of Equation (3) we get

$$\left.\begin{aligned} I(L_{11})I(L_{21}) - I(L_{12})I(L_{22}) &= I(L_{11}L_{21}) - I(L_{12}L_{22}), \\ I(L_{11})I(L_{22}) + I(L_{12})I(L_{21}) &= I(L_{11}L_{22}) + I(L_{12}L_{21}). \end{aligned}\right\} \tag{4}$$

Therefore, the theorem follows if we show that Equation (4) holds. We will demonstrate that $I(L_{11})I(L_{21}) = I(L_{11}L_{21})$ for each $f \epsilon Y'$; the other three cases are similar.

Let $f \epsilon Y'$ be fixed. Since $|L_{11}| \leq 1$ and $|L_{21}| \leq 1$, there are simple functions S_n and T_k so that $S_n \to L_{11}$, $T_k \to L_{21}$, $|S_n| \leq 1$ for all n, and $|T_k| \leq 1$ for all k. For each n(k) there is one and only one $A_n(B_k)$ in $\mu^{-1}(1)$ on which $S_n(T_k)$ is constant because μ is a zero-one measure. Let $D = \left(\bigcap_{n=1}^{\infty} A_n\right) \cap \left(\bigcap_{k=1}^{\infty} B_k\right)$. Then D is non-empty because $\mu^{-1}(1)$ is closed under countable intersections. Let $a \epsilon D$. Then, by the dominated convergence theorem for the Lebesgue integral, we have $L_{11}(a) \leftarrow S_n(a) = I(S_n) \to I(L_{11})$ and $L_{21}(a) \leftarrow T_k(a) = I(T_k) \to I(L_{21})$. Hence $L_{11}(a) = I(L_{11})$ and $L_{21}(a) = I(L_{21})$. But also $(S_n)(T_k) \to (L_{11})(L_{21})$. The dominated convergence theorem implies that $L_{11}(a)L_{21}(a) \leftarrow S_n(a)T_k(a) = I(S_nT_k) \to I(L_{11}L_{21})$. Therefore $I(L_{11})I(L_{21}) = I(L_{11}L_{21})$. □

<u>Corollary 1 to Theorem 2</u>. If μ is a zero-one measure, $\phi \epsilon D(\hat{I})$, and if S is a countable subset of Y', then there is an $a \epsilon X$ such that $I\left(f(\phi)\right) = f\left(\phi(a)\right)$ for every $f \epsilon S$.

<u>Proof</u>. Obvious extension of part (2) in the proof of Theorem 2. □

<u>Corollary 2 to Theorem 2</u>. If μ is a zero-one measure, then $\hat{I}(\phi_1 \oplus \phi_2) = \hat{I}(\phi_1) \oplus \hat{I}(\phi_2)$ on $D_o(\hat{I})$.

<u>Proof</u>. Let $\hat{I}(\phi_i) = y_i$ for i=1,2. Then by Theorem 2 $\hat{I}(\phi_1 \oplus \phi_2) = \hat{I}(\phi_1)\hat{I}(\phi_2) = F_{y_1}F_{y_2} = F_{y_1 \oplus y_2}$. Hence $y_1 \oplus y_2 = \hat{I}(\phi_1 \oplus \phi_2)$. □

Let $A \epsilon \mathcal{A}$. If $\phi \epsilon (X,Y)$ and ϕ is constant on A and constant on A^c, then ϕ is a <u>two-valued simple function</u>.

<u>Corollary 3 to Theorem 2</u>. If $\hat{I}(\phi_1 \oplus \phi_2) = \hat{I}(\phi_1)\hat{I}(\phi_2)$ for all two-valued simple functions, then μ is a zero-one measure.

<u>Proof</u>. Part (1) in the proof of Theorem 2. □

Let $A \epsilon \mathcal{A}$ and assume that $y \epsilon Y$ and $y \neq \theta$. If $\phi = y$ on A and $\phi = \theta$ on A^c, then ϕ is a <u>characteristic function</u>.

<u>Corollary 4 to Theorem 2</u>. For each $\phi \epsilon D(\hat{I})$, let $F_\phi = \hat{I}(\phi)$. If $F_\phi(f_1 f_2) = F_\phi(f_1)F_\phi(f_2)$ on Y' for all characteristic functions ϕ, then μ is a zero-one measure. If μ is a zero-one measure, then $F_\phi(f_1 f_2) = F_\phi(f_1)F_\phi(f_2)$ on Y' for each $\phi \epsilon D(\hat{I})$.

<u>Proof</u>

(1) Let μ be a zero-one measure. The proof that $F_\phi(f_1 f_2) = I\left[f_1(\phi)f_2(\phi)\right] = I\left[f_1(\phi)\right]I\left[f_2(\phi)\right] = F_\phi(f_1)F_\phi(f_2)$ on Y' for each $\phi \epsilon D(\hat{I})$ is identical to part (2) in the proof of Theorem 2.

(2) Assume that $F_\phi(f_1 f_2) = F_\phi(f_1)F_\phi(f_2)$ on Y' for each characteristic function ϕ. If we let $f_1 = f_2 = \mathit{1}$ we again obtain $\mu(X)$ is zero or one.

(a) Assume that $\mu(X) = 1$ and let $A \epsilon \mathcal{A}$. We will show that $\mu(A)$ is zero or one.

Let $y \epsilon Y$, $y \neq \theta$, and let ϕ map A onto y and map A^c onto θ. From $I\left[f_1(\phi)f_2(\phi)\right] = I\left[f_1(\phi)\right]I\left[f_2(\phi)\right]$ holding on Y' we get $\mu(A)\left[\mu(A)-1\right]\Gamma = 0$, where $\Gamma = f_1(y)f_2(y) - f_1(y) - f_2(y) + 1$, because $\mu(A^c) = 1 - \mu(A)$ and $f_i(\theta) = 1$ for $i=1,2$. We now show that Γ cannot be zero for all f_1, f_2 in Y', implying that $\mu(A)$ is zero or one.

Since Y' separates points in Y, there is a f in Y' so that $f(y) \neq 1$. If $f_1 = f_2 = f$, then $\Gamma = \left(f(y)\right)^2 - 2f(y) + 1 = \left(f(y)-1\right)^2 \neq 0$.

(b) Assume that $\mu(X) = 0$ and let $A \in \mathcal{A}$. We will show that $\mu(A) = 0$. This is a contradiction because we have assumed that μ is not identically zero.

Define ϕ as in (a) above. From $I\left(f_1(\phi)f_2(\phi)\right) = I\left(f_1(\phi)\right)I\left(f_2(\phi)\right)$ holding on Y' we get $\mu(A)\left(\mu(A)\Gamma+1-f_1(y)f_2(y)\right) = 0$ because $\mu(A^c) = -\mu(A)$. Since Y' separates points in Y, there is a $f \in Y'$ such that $f(y) = a+bi$ and $a \neq 1$. If $f_1 = f$ and $f_2 = \bar{f}$, then we get $\left(\mu(A)\right)^2\left(2-\overline{f(y)}-f(y)\right) = 0$ because $f(y)\overline{f(y)} = 1$. But $2 - (a-bi) - (a+bi) \neq 0$ because $a \neq 1$. Hence $\mu(A) = 0$. $\square$

Corollary 5 to Theorem 2. If $\psi \in D(I)$ and if μ is a zero-one measure, then $I(\psi) = \psi(a)$ for some $a \in X$.

Proof. From part (2) in the proof of Theorem 2. $\square$

Theorem 3. If $D(\hat{I}) = D_o(\hat{I})$, then μ is a zero-one measure.

Proof

(1) Let $\mu = m_1 + m_2 i = (\mu_1 - \mu_2) + (\mu_3 - \mu_4)i$ where the μ_i are finite measures and the m_i are finite signed measures (see Appendix B). Let $\phi \in D(\hat{I})$, then there is a $y \in Y$ such that $f(y) = I\left(f(\phi)\right)$ for every

$f \epsilon Y'$. Hence $\mu(X) = I\left(1(\phi)\right) = 1(y) = 1$.

(2) Let $\phi \epsilon D(\hat{I})$ be fixed. For each $f \epsilon Y'$ we will write $\left(f(\phi)\right)_1$ for the real part of $f(\phi)$ and $\left(f(\phi)\right)_2$ for the imaginary part of $f(\phi)$. For $j = 1,2,3,4$, $I(\psi d\mu_j)$ denotes the Lebesgue integral of real-valued ψ over X with respect to μ_j. Let $ij = I\left(\left(f(\phi)\right)_i d\mu_j\right)$ for $i = 1,2$ and for $j = 1,2,3,4$. Then (see Appendix B) $I\left(f(\phi)\right) = I\left(f(\phi)dm_1\right) + iI\left(f(\phi)dm_2\right)$ where $I\left(f(\phi)dm_1\right) = (11-12) + (21-22)i$ and $I\left(f(\phi)dm_2\right) = (13-14) + (23-24)i$.

We will now show that $I\left(f(\phi)dm_2\right) = 0$ for every $f \epsilon Y'$. There is a $y \epsilon Y$ such that $f(y) = I\left(f(\phi)\right)$ for each $f \epsilon Y'$. If $\bar{f}(y) = \overline{f(y)}$ for each $y \epsilon Y$, then $\bar{f}$ is in Y' if f is in Y'. Therefore $I\left(\bar{f}(\phi)\right) = I\left(\overline{f(\phi)}\right) = \bar{f}(y) = \overline{f(y)} = \overline{I\left(f(\phi)\right)}$ for each $f \epsilon Y'$. That is

$$I\left(\overline{f(\phi)}\right) = \overline{I\left(f(\phi)\right)} \tag{5}$$

for each $f \epsilon Y'$. If we equate the real and imaginary parts of Equation (5), using Equation (1) in Appendix B, we get $11-12-23+24 = 11-12+23-24$ and $14+22-13-21 = 13-14-21+22$. Hence $23 = 24$ and $13 = 14$. Therefore $I\left(f(\phi)dm_2\right) = 0$ for each $f \epsilon Y'$.

(3) Next we show that $m_2(A) = 0$ for every $A \epsilon \mathcal{A}$. Let $A \epsilon \mathcal{A}$. Now $\mu(X) = 1$ implies that $m_2(X) = 0$. Hence $m_2(A^C) = -m_2(A)$.

Let $y \epsilon Y$, $y \neq \theta$, and let $f \epsilon Y'$ such that $f(y) = a+bi$ and $a \neq 1$. Define $\phi = y$ on A and $\phi = \theta$ on A^C. Surely $\phi \epsilon D(\hat{I})$. From part (2) above we have $0 = I\left(f(\phi)dm_2\right) = (a+bi)m_2(A) + m_2(A^C) = (a-1)m_2(A) + bm_2(A)i$. Therefore $m_2(A) = 0$ because $a \neq 1$. Hence $\mu = m_1$ because m_2 is identically zero.

(4) Finally we show that $m_1(A)$ is zero or one for every $A\epsilon\mathcal{A}$. It follows that μ is a zero-one measure.

Let $A\epsilon\mathcal{A}$. Select y and ϕ as in part (3) above. f is a fixed member of Y' such that $f(y) = a + bi$ and $a\neq 1$. There is a $t\epsilon Y$ so that $f(t) = I(f(\phi))$ because $\phi\epsilon D_o(\hat{I})$. Therefore $1 = |f(t)| = |I(f(\phi))| = |(a+bi)m_1(A) + m_1(A^c)| = |((a-1)m_1(A) + 1) + (bm_1(A))i|$ because $m_2 = 0$ and $m_1(A^c) = 1 - m_1(A)$. This implies that $(a-1)^2(m_1(A))^2 + 2(a-1)m_1(A) + 1 + b^2(m_1(A))^2 = 1$, or $(a-1)m_1(A)(1-m_1(A)) = 0$ because $a^2 + b^2 = 1$. Hence $m_1(A)$ is zero or one because $a\neq 1$. □

Corollary 1 to Theorem 3. If all characteristic functions belong to $D_o(\hat{I})$, then μ is a zero-one measure. If μ is a zero-one measure, then all simple functions belong to $D_o(\hat{I})$.

Proof

(a) The first part follows from the proof of Theorem 3.

(b) Let μ be a zero-one measure and let ϕ be a simple function in (X,Y). There is one and only one set $A\epsilon\mathcal{A}$, with measure one, on which ϕ is constant because μ is a zero-one measure. Let $x\epsilon A$ and $y = \phi(x)$. Then $I(f(\phi)) = f(y)\mu(A) = f(y)$ for every $f\epsilon Y'$. Hence $\phi\epsilon D_o(\hat{I})$. □

Remarks on Theorem 3 and Its Corollary

(a) From Theorem 3 we see that: (1) if $D_o(\hat{I})$ is non-empty, then $\mu(X) = 1$; (2) if $\phi\epsilon D_o(\hat{I})$, then $I(f(\phi)dm_2) = 0$ for every $f\epsilon Y'$; and (3) if $\phi\epsilon D_o(\hat{I})$, then $I(\overline{f(\phi)}) = \overline{I(f(\phi))}$ for every $f\epsilon Y'$.

(b) Let Y be an arbitrary non-empty set and let Y' be a non-empty subset of $(Y,C_\cdot)$. From part (b) of Corollary 1 to Theorem 3 we see that all the simple functions will be in $D_o(\hat{I})$ whenever μ is a zero-one measure.

The next theorem gives a partial converse to Theorem 3. We do not know if the converse of Theorem 3 is true or false. The following definitions and lemmas will be used in Theorem 4.

If K is a set, then card (K) denotes the cardinality of K. In part of Theorem 4 we will assume that the card (X) is accessible. See Sikorski ([27], p.86) for the definition of an accessible cardinal. This assumption is not too restrictive an assumption because $a_0, a_1, a_2, \cdots$ are all accessible cardinals if $a_0 = \text{card}\{1,2,3,\cdots\}$ and $a_{n+1} = 2^{a_n}$ for $n = 0,1,2,\cdots$ ([27], p. 86).

Let $\mathcal{B}$ be the minimal σ-algebra over the open sets in Y and let $\mathcal{S}$ be the minimal σ-ring over the compact subsets of Y. If $\mu = (\mu_1 - \mu_2) + (\mu_3 - \mu_4)i$, then for part (b) in the next definition we will assume that at least one of the μ_j is complete (say μ_1). The reason for this assumption is that if measurable complex-valued $h_n \to h$ a.e. (μ_1) or if measurable complex-valued $k = h$ a.e. (μ_1), then we can conclude that h is also measurable.

<u>Definition 2</u>

(a) $D_1 = \{\phi \in (X,Y) \mid \bar{\phi}^{-1}(\mathcal{B}) \subset \mathcal{A}$ and $\phi^{-1}(\mathcal{S}) \cap \mu^{-1}(1) \neq \emptyset\}$.

(b) $D_2 = \{\phi \in (X,Y) \mid \phi$ is the a.e. (μ_1) limit of simple functions$\}$ if μ_1 is complete.

(c) $D_3 = \{\phi \in D(\hat{I}) \mid \hat{I}(\phi)$ is continuous$\}$.

(d) $D_4 = \{\phi\epsilon(X,Y) | f(\phi)$ is measurable for each f in Y' and there is a compact subset C of Y and an A in $\mu^{-1}(1)$ such that $\phi(A)\subset C\}$.

<u>Lemma 2</u>. For $i = 1,2,4, D_i \subset D(\hat{I})$.

<u>Proof</u>

(a) Let $\phi\epsilon D_1$, $f\epsilon Y'$, and $f = f_1 + f_2 i$ where f_1 and f_2 are the real and imaginary parts of f, respectively. Now the real and imaginary parts of $f(\phi)$ are $f_1(\phi)$ and $f_2(\phi)$, respectively. If we show that each $f_i(\phi)$ is measurable, it follows that $\phi\epsilon D(\hat{I})$ because $|f_i(\phi)| \le 1$ for $i = 1,2$ and $|\mu(X)| < +\infty$.

Each f_i is measurable with respect to $\mathcal{B}$ because each f_i is continuous. Therefore each $f_i(\phi)$ is measurable.

(b) Let $\phi\epsilon D_2$ and let $f\epsilon Y'$. If ϕ_n is a sequence of simple functions converging to ϕ a.e. (μ_1), then $f(\phi_n)$ is a sequence of simple functions converging to $f(\phi)$ a.e. (μ_1) because f is continuous. Therefore each $f_i(\phi)$ is the a.e. (μ_1) limit of simple functions. This implies that each $f_i(\phi)$ is measurable because μ_1 is complete. Therefore $\phi\epsilon D(\hat{I})$.

(c) Let $\phi\epsilon D_4$. It follows that $\phi\epsilon D(\hat{I})$ because $|f_i(\phi)| \le 1$ for $i=1,2$ and each f in Y', and $|\mu(X)| < +\infty$. □

<u>Lemma 3</u>. Let μ be a zero-one measure and $F_\phi = \hat{I}(\phi)$ for $\phi\epsilon D(\hat{I})$. Then F_ϕ is a bounded, not identically zero, homomorphism of Y' into $C_\bullet$.

<u>Proof</u>

(a) $F_\phi(1) = I[1(\phi)] = \mu(X) = 1$. Hence F_ϕ is not identically zero.

(b) Let $f \in Y'$. Then $|F_\phi(f)| = |I(f(\phi))| \le |I(f_1(\phi))| + |I(f_2(\phi))| \le I(|f_1(\phi)|) + I(|f_2(\phi)|) \le I(1) + I(1) = 2\mu(X) = 2$ where f_1 and f_2 are the real and imaginary parts of f, respectively. Hence F_ϕ is bounded.

(c) Corollary 4 to Theorem 2 implies that F_ϕ is a homomorphism of Y' into $C_\bullet$. □

Remarks on Lemma 3

Let μ be a zero-one measure and $\phi \in D(\hat{I})$.

(1) If F_ϕ is continuous, then Lemma 3 implies that $F_\phi \in Y''$. Therefore, by Theorem 1, there is a $y \in Y$ such that $\lambda(y) = F_\phi$. Hence $y = \hat{I}(\phi)$ and $\phi \in D_o(\hat{I})$. This fact will be used many times in Theorem 4.

(2) We can easily see, by a proof similar to that of Lemma 1, that F_ϕ maps Y' into T.

Theorem 4. Let μ be a zero-one measure.

(1) $D(\hat{I}) = D_o(\hat{I})$ if one of the following conditions hold:

(a) $A = \{A | A \subset X\}$ and card (X) is accessible;

(b) $\cap\{A | A \in \mu^{-1}(1)\} \ne \emptyset$;

(c) X is a metric space, A is the Borel field, and card (X) is accessible;

(d) Y is compact;

(e) Y is second countable;

(f) Y' is first countable;

(g) Y' is separable;

(h) Y' is countable.

(2) (a) If $\phi \epsilon D_i$ for any i=1,3,4, then $\phi \epsilon D_o(\hat{I})$.

(b) If $\phi \epsilon D_2$ and μ is complete, then $\phi \epsilon D_o(\hat{I})$.

Proof

(1) (i) Here we will consider conditions (a), (b), and (c) on X. Sikorski ([27], pp.88-90) shows that (a) implies (b) and (c) implies (b). Let $a \epsilon \cap \{A | A \epsilon \mu^{-1}(1)\}$ and let $\phi \epsilon D(\hat{I})$ be fixed.

If $f \epsilon Y'$, we write $f = f_1 + f_2 i$. For each $f \epsilon Y'$ there are two sequences of simple functions $S_n(f_j)$ so that: (a) $S_n(f_j) \to f_j(\phi)$ for j=1,2; and (b) $|S_n(f_j)| \leq 1$ for all n and for j=1,2. For each $f \epsilon Y'$ and each n there is one and only one $A_n(f_j)$ in $\mu^{-1}(1)$ on which $S_n(f_j)$ is constant because μ is a zero-one measure. Now $a \epsilon \cap \{A_n(f_j) | f \epsilon Y', j=1,2, n=1,2,3,\cdots\}$. Therefore we have, by the dominated convergence theorem, that $f(\phi(a)) = f_1(\phi(a)) + f_2(\phi(a))i \leftarrow S_n(f_1)(a) + S_n(f_2)(a)i = I(S_n(f_1)) + I(S_n(f_2))i \to I(f_1(\phi)) + I(f_2(\phi))i = I(f(\phi))$ for each $f \epsilon Y'$. Hence $\phi(a) = \hat{I}(\phi)$ and $\phi \epsilon D_o(\hat{I})$.

(ii) Assume that Y is compact. Then Y' is discrete ([21], p.362). Therefore $F_\phi = \hat{I}(\phi)$ is automatically continuous for each $\phi \epsilon D(\hat{I})$. Lemma 3 and Theorem 1 imply that $D(\hat{I}) = D_o(\hat{I})$.

(iii) Assume that Y is second countable. Then Y' is also second countable ([21], p.381). It suffices to show that F_ϕ is continuous for each $\phi \epsilon D(\hat{I})$.

Let $\phi \epsilon D(\hat{I})$ and let f_n be a <u>sequence</u> in Y' converging to f in Y'. Then f_n converges to f pointwise ([20], p.222). Therefore

$f_n(\phi)$ converges to $f(\phi)$ pointwise. Let $h_n(h)$ be the real part of $f_n(\phi)(f(\phi))$ and $g_n(g)$ be the imaginary part of $f_n(\phi)(f(\phi))$. Since $|h_n| \leq 1$ and $|g_n| \leq 1$ with $I(1) = 1$, the dominated convergence theorem implies that $F_\phi(f_n) = I(f_n(\phi)) = I(h_n) + I(g_n)i \to I(h) + I(g)i = I(f(\phi)) = F_\phi(f)$. Hence F_ϕ is continuous.

(iv) Assume that Y' is first countable. In order to show that F_ϕ is continuous it suffices to show that $F_\phi(f_n) \to F_\phi(f)$ whenever f_n is a <u>sequence</u> in Y' converging to f in Y'. It follows that each F_ϕ is continuous by part (iii) above.

(v) Assume that Y' is separable. Let H be a countably dense subset of Y' and let <u>net</u> f_β in Y' converge to f in Y'. If $F_\phi(f_\beta) \to F_\phi(f)$, then F_ϕ is continuous and $\phi \epsilon D_o(\hat{I})$. Let $\phi \epsilon D(\hat{I})$ be fixed.

For each $\beta \epsilon D$, the directed set of the net f_β, there is a directed set E_β and a net $f(\beta,\alpha)$, $\alpha \epsilon E_\beta$, in H such that $\lim\limits_{\alpha \epsilon E_\beta} f(\beta,\alpha) = f_\beta$.

We now show that for each β in D, $\lim\limits_{\alpha \epsilon E_\beta} F_\phi(f(\beta,\alpha)) = F_\phi(f_\beta)$. Let β in D be fixed. If $A = \{f_\beta\} \cup \{f(\beta,\alpha) | \alpha \epsilon E_\beta\}$, then A is countable. Corollary 1 to Theorem 2 implies that there is an $a \epsilon X$ such that $I(f(\phi)) = f(\phi(a))$ for every $f \epsilon A$. Therefore $\lim\limits_{\alpha \epsilon E_\beta} F_\phi(f(\beta,\alpha)) = \lim\limits_{\alpha \epsilon E_\beta} I(f(\beta,\alpha)(\phi)) = \lim\limits_{\alpha \epsilon E_\beta} f(\beta,\alpha)(\phi(a)) = f_\beta(\phi(a)) = I(f_\beta(\phi)) = F_\phi(f_\beta)$ because $f(\beta,\alpha)$ converges to f_β pointwise ([20], p.222).

Now let $U = H \cup \{f\}$. Corollary 1 to Theorem 2 implies that there is an $a \epsilon X$ such that $I(f(\phi)) = f(\phi(a))$ for every $f \epsilon U$ because

U is countable. Therefore $\lim_{\beta\in D} F_\phi(f_\beta) = \lim_{\beta\in D}\left(\lim_{\alpha\in E_\beta} F_\phi\big(f(\beta,\alpha)\big)\right) =$

$\lim_{\beta\in D}\left(\lim_{\alpha\in E_\beta} I\big(f(\beta,\alpha)(\phi)\big)\right) = \lim_{\beta\in D}\left(\lim_{\alpha\in E_\beta} f(\beta,\alpha)\big(\phi(a)\big)\right) = \lim_{\beta\in D} f_\beta\big(\phi(a)\big) =$

$f\big(\phi(a)\big) = I\big(f(\phi)\big) = F_\phi(f)$. Therefore F_ϕ is continuous.

(vi) Now assume that Y' is countable. This is quite possible because if Y is compact, then Y is metrizable if and only if Y' is countable ([21], p.382). Corollary 1 to Theorem 2 says that if $\phi\epsilon D(\hat{I})$, then there is an $a\epsilon X$ so that $I\big(f(\phi)\big) = f\big(\phi(a)\big)$ for every $f\epsilon Y'$. Therefore $D(\hat{I}) = D_o(\hat{I})$.

(2) (a) (i) If $\phi\epsilon D_3$, then Lemma 3 and Theorem 1 imply that $\phi\epsilon D_o(\hat{I})$.

(ii) We will show that $D_1\subset D_4$. So, when it is shown that $D_4\subset D_o(\hat{I})$, it will follow that $D_1\subset D_o(\hat{I})$.

We say that a subset E of Y is <u>σ-bounded</u> if and only if there is a sequence C_n of compact subsets of Y such that $E\subset \bigcup_{n=1}^{\infty} C_n$. Each $E\epsilon S$ is σ-bounded ([28], p.219).

Let $\phi\epsilon D_1$ and let $A\epsilon\phi^{-1}(S)\cap\mu^{-1}(1)$. Choose $B\epsilon S$ so that $\phi^{-1}(B) = A$. There is a sequence C_n of compact subsets of Y so that $B\subset \bigcup_{n=1}^{\infty} C_n$. It follows that at least one of the $\phi^{-1}(C_n)$ has measure one. For assume that each $\phi^{-1}(C_n)$ has measure zero. Then, since $A\subset \bigcup_{n=1}^{\infty}\phi^{-1}(C_n)$, we have $1 = \mu(A) \leq \sum_{n=1}^{\infty}\mu\big(\phi^{-1}(C_n)\big) = 0$. Assume that $\phi^{-1}(C_{n_o})$ has measure one. Then $\phi^{-1}(C_{n_o})\epsilon\mu^{-1}(1)$ and $\phi\big(\phi^{-1}(C_{n_o})\big)\subset C_{n_o}$. It follows that ϕ is in D_4 if $f(\phi)$ is measurable for every f in Y'. Lemma 2 implies that $\phi\epsilon D(\hat{I})$, so that $f(\phi)$ is measurable for every f in Y'.

(iii) Let $\phi \in D_4$. We will show that ϕ is in $D_o(\hat{I})$. It suffices to show that $F_\phi(f_\beta) \to F_\phi(f)$ whenever net f_β in Y' converges to f in Y'.

We first show that $F_\phi(f_\beta) \to F_\phi(\mathit{1}) = 1$ whenever net f_β in Y' converges to $\mathit{1}$. Let C be a compact subset of Y and let $A \in \mu^{-1}(1)$ such that $\phi(A) \subset C$. Given $\epsilon > 0$ there is a β_o such that $\beta \geq \beta_o$ implies $f_\beta(C) \subset \{z \mid |z-1| < \epsilon\}$ because $f_\beta \to \mathit{1}$ with respect to the compact open topology. Therefore $|F_\phi(f_\beta) - 1| = |I(f_\beta(\phi)) - 1| = |I(f_\beta(\phi),A) - 1| = |I(f_\beta(\phi) - 1,A)| \leq I(|f_\beta(\phi) - 1|,A) \leq \epsilon$ for $\beta \geq \beta_o^*$.

Let net f_β in Y' converge to f in Y'. Then $f_\beta f^{-1} = f_\beta \bar{f} \to 1$ because Y' is a topological group. $F_\phi(f_\beta \bar{f}) = F_\phi(f_\beta)F_\phi(\bar{f})$ by Lemma 3. Therefore $F_\phi(f_\beta)F_\phi(\bar{f}) \to 1$. But $1 = F_\phi(\mathit{1}) = F_\phi(\bar{f}f) = F_\phi(\bar{f})F_\phi(f)$. Hence $F_\phi(f_\beta) = F_\phi(f_\beta)F_\phi(\bar{f})F_\phi(f) \to F_\phi(f)$ and F_ϕ is continuous.

(b) Let $\phi \in D_2$. The measure μ_1 must be equal to μ because μ is a real-valued measure. Therefore there are simple functions ϕ_n such that $\phi_n \to \phi$ on B and $\mu(B^c) = 0$. For each n there is one and only one A_n in $\mu^{-1}(1)$ on which ϕ_n is constant. Let a belong to $\left(\bigcap_{n=1}^{\infty} A_n\right) \cap B$, which is nonempty because $\mu^{-1}(1)$ is closed under countable intersections.

Let $f \in Y'$ and let $f = f_1 + f_2 i$. Each $f_i(\phi)$ is measurable because μ is complete and $f_i(\phi_n) \to f_i(\phi)$ a.e. for i=1,2. Note

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*Rudin ([30], p.25) shows that $|I(g,A)| \leq I(|g|,A)$ for g complex-valued and μ real-valued.

that $|f_i(\phi_n)| \le 1$ for all n and for i=1,2. Then, by the dominated convergence theorem, we have $f(\phi(a)) \leftarrow f(\phi_n(a)) = I(f(\phi_n)) = I(f_1(\phi_n)) + I(f_2(\phi_n))i \to I(f_1(\phi)) + I(f_2(\phi))i = I(f(\phi))$. Hence $\phi(a) = \hat{I}(\phi)$ and $\phi \in D_o(\hat{I})$. □

Assume that μ is a zero-one measure and that $B = \cap\{A \mid A \in \mu^{-1}(1)\}$. We will now give examples of B empty; B a point and in $\mathcal{A}$; B not empty, not a point, and in $\mathcal{A}$; B a point and not in $\mathcal{A}$; and B not empty, not a point, and not in $\mathcal{A}$. Let X = [0,1].

Example 1. Let $N = \{A \subset X \mid A \text{ is countable}\}$, $N' = \{A^c \mid A \in N\}$, and let $\mathcal{A} = N \cup N'$. $\mathcal{A}$ is a σ-algebra. If $\mu(N) = 0$ and $\mu(N') = 1$, then B is empty.

Example 2. Let K be a non-empty subset of X, $N = \{A \subset X \mid K \subset A\}$, $N' = \{A^c \mid A \in N\}$, and let $\mathcal{A} = N \cup N'$. If $\mu(N) = 1$ and $\mu(N') = 0$, then B = K in $\mathcal{A}$. If K is a point set, then so is B. If we had defined $N = \{A \subset X \mid K \subset A \text{ and } K \neq A\}$, then B = K not in $\mathcal{A}$.

The last theorem in this chapter shows that $\hat{I}$ is faithful (Definition 6 of Chapter III) if and only if μ is a zero-one measure. We must first define the category $\mathcal{C}$.

Let $\mathcal{O}$ be the collection of all locally compact, Hausdorff, Abelian topological groups together with $C_\bullet$. Note that $C_\bullet$ is the range space of the functionals and it is not a group. If Y, Y_1 are in $\mathcal{O}$ and Y_1 is not equal to $C_\bullet$, then we define the morphisms as follows:

(a) $\hom(Y,Y_1)$ is the set of all continuous homomorphisms on Y into Y_1;

(b) $\hom(Y_1,C_\bullet) = Y_1'$ is the set of all bounded, not identically zero, continuous homomorphisms on Y_1 into $C_\bullet$;

(c) $\hom(C_\bullet,C_\bullet)$ is the identity map together with $\mathit{1}$, where $\mathit{1}$ maps $C_\bullet$ onto 1. Note that $\hom(C_\bullet,Y_1)$ consists only of the function which maps $C_\bullet$ onto the identity of Y_1. $\mathcal{O}$ together with these morphisms forms the category $\mathcal{C}$. See Chapter III for the definition of a category.

For each object Y, let $D(\hat{I},Y)$ be the domain of $\hat{I}$ corresponding to the set of functionals $Y' = \hom(Y,C_\bullet)$. Category $\mathcal{C}$ is the type of category that was considered in Chapter III if: (a) $C_\bullet$ is an object; (b) $Y' = \hom(Y,C_\bullet)$ is non-empty for each object Y; and (c) $D(\hat{I},Y)$ is non-empty for each object Y. For each object Y, each constant function on X into Y belongs to $D(\hat{I},Y)$ because $|\mu(X)| < +\infty$. Therefore, the category $\mathcal{C}$ defined above satisfies these three conditions.

We now show that $\hom(C_\bullet,C_\bullet)$ is the set of all bounded, not identically zero, continuous homomorphisms on $C_\bullet$ into $C_\bullet$ together with the identity map. If $\mathcal{C}'$ is any category, then the identity map must belong to $\hom(Y,Y)$ for any object Y. Let f be a bounded, not identically zero, continuous homomorphism of $C_\bullet$ into $C_\bullet$. Then $f(0)$ is either zero or one. If $f(0) = 1$, then $f = \mathit{1}$. If $f(0) = 0$, then f is not continuous because f maps $C_\bullet^+$ ($C_\bullet$ without zero) into T (Lemma 1).

Theorem 5

(a) Let $U = C_+$. $\hat{I}$ is faithful on $D(I)$ with respect to $\{U,U'\}$ if and only if μ is a zero-one measure.

(b) Let $U = T$. $\hat{I}$ is faithful on $D(\hat{I},U)$ with respect to $\{U,U'\}$ if and only if μ is a zero-one measure.

(c) Let $U = C_{\bullet}$. $\hat{I}$ is faithful on $D(\hat{I},U) = D(I)$ with respect to $\{U,U'\}$ if and only if $\mu(X) = 1$.

(d) Let $U = C_{\bullet}^{+}$. $\hat{I}$ is faithful on $N(I) = \{\phi\epsilon D(I) | \phi(x)$ is not zero for each x in X$\}$ with respect to $\{U,U'\}$ if and only if μ is a zero-one measure.

Proof

(a) (1) Assume that $\hat{I}$ is faithful. Let ϕ be a characteristic function in (X,U). Surely $\phi\epsilon D(I)$. This means that $\lambda^{-1}(\hat{I}(\phi)) = I(\phi)$. Therefore $\phi\epsilon D_o(\hat{I},U)$. Corollary 1 to Theorem 3 implies that μ is a zero-one measure.

(2) Assume that μ is a zero-one measure. If $\phi\epsilon D(I)$, then surely $\phi\epsilon D(\hat{I},U)$ because each $f\epsilon U'$ is continuous and bounded, and $\mu(X) = 1$. Therefore $D(I)$ is a subset of $D(\hat{I},U)$. If we can show that $I(f(\phi)) = f(I(\phi))$ for every $f\epsilon U'$, for each $\phi\epsilon D(I)$, then it follows that $\hat{I}(\phi) = I(\phi)$ on $D(I)$.

Now $U' = \{f(x+iy) = \exp(i(ax+by)) | a$ and b real$\}$ ([21], p.368), so U' is separable with $H = \{f(x+iy)\epsilon U' | a$ and b are rational$\}$ a countably dense subset. Theorem 4 implies that $D(\hat{I},U) = D_o(\hat{I},U)$. Therefore, if $\phi\epsilon D(I)$, then there is a y_ϕ in U such that $I(f(\phi)) = f(y_\phi)$ for every f in U' and $\hat{I}(\phi) = y_\phi$. We must show that $y_\phi = I(\phi)$.

We will show that $F_\phi(f) = I(f(\phi)) = f(y_\phi = I(\phi))$ for each $f\epsilon H$ and each $\phi\epsilon D(I)$. It then follows that $I(f(\phi)) = f(I(\phi))$ for every $f\epsilon U'$ and each $\phi\epsilon D(I)$ because if f is in U' and not in H, there is a sequence f_n in H converging to f. Then $I[f(\phi)] = F_\phi(f) \leftarrow F_\phi(f_n) = I[f_n(\phi)] = f_n[I(\phi)] = f_n(y_\phi) \rightarrow f(y_\phi) = f[I(\phi)]$ because F_ϕ is continuous (Theorem 4) and f_n converges to f pointwise.

Let $\phi \epsilon D(I)$ be fixed and let ϕ_1, ϕ_2 denote the real and imaginary parts of ϕ, respectively. Then $I[f(\phi)] = f[I(\phi)]$ for every $f \epsilon H$ if and only if $I[\cos(a\phi_1+b\phi_2)] + I[\sin(a\phi_1+b\phi_2)]i = \cos[aI(\phi_1) + bI(\phi_2)] + \sin[aI(\phi_1) + bI(\phi_2)]i$ for all rationals a and b.

We will show that there is a $\bar{x}\epsilon X$ such that $I[\cos(a\phi_1 + b\phi_2)] = \cos[a\phi_1(\bar{x}) + b\phi_2(\bar{x})]$, $I[\sin(a\phi_1 + b\phi_2)] = \sin[a\phi_1(\bar{x}) + b\phi_2(\bar{x})]$ for all rationals a and b, and $I(\phi_1) = \phi_1(\bar{x})$, $I(\phi_2) = \phi_2(\bar{x})$. It then follows that $I[f(\phi)] = f[I(\phi)]$ for every f in H.

Let $\phi_1^+(\phi_2^+)$ be the positive part of $\phi_1(\phi_2)$ and $\phi_1^-(\phi_2^-)$ be the negative part. There are four sequences $J(n,i)$, $i=1,\cdots,4$, of non-negative simple functions such that $J(n,1) \uparrow \phi_1^+$, $J(n,2) \uparrow \phi_1^-$, $J(n,3) \uparrow \phi_2^+$, and $J(n,4) \uparrow \phi_2^-$. There are two countable collections of simple functions $S(n,i)$ and $T(n,i)$ such that: (a) for each pair of rationals a and b there is an i_o so that $S(n,i_o) \rightarrow \cos(a\phi_1 + b\phi_2)$; (b) for each pair of rationals a and b there is an i_o so that $T(n,i_o) \rightarrow \sin(a\phi_1 + b\phi_2)$; (c) $|S(n,i)| \leq 1$ and $|T(n,i)| \leq 1$ for all i and n. For each $i=1,2,3,4$ and each n there is one and only one $A(n,i)$ in $\mu^{-1}(1)$ on which $J(n,i)$ is constant. Also, for each i and n there is one and only one $B(n,i)$ $[D(n,i)]$ in $\mu^{-1}(1)$ on which $S(n,i)[T(n,i)]$ is constant. Let $\bar{x}$ be in the intersection of all the $A(n,i)$, $B(n,i)$, and $D(n,i)$. Such a point exists because $\mu^{-1}(1)$ is closed under countable intersections. The monotone convergence theorem implies that $\phi_1^+(\bar{x}) \leftarrow J(n,1)(\bar{x}) = I(J(n,1)) \rightarrow I(\phi_1^+)$. Similarly we get $\phi_1^-(\bar{x}) = I(\phi_1^-)$, $\phi_2^+(\bar{x}) = I(\phi_2^+)$, and $\phi_2^-(\bar{x}) = I(\phi_2^-)$. Therefore $I(\phi_1) = \phi_1(\bar{x})$ and $I(\phi_2) = \phi_2(\bar{x})$. The dominated convergence theorem implies that $\cos(a\phi_1(\bar{x}) + b\phi_2(\bar{x})) \leftarrow S(n,i_o)(\bar{x}) = I(S(n,i_o)) \rightarrow I(\cos(a\phi_1 + b\phi_2))$. Similarly we see that $\sin(a\phi_1(\bar{x}) + b\phi_2(\bar{x})) = I(\sin(a\phi_1 + b\phi_2))$ for each rational a and b.

(b) (1) Let $\hat{I}$ be faithful on $D(\hat{I},U)$ when $U = T$. If $\phi \epsilon D(\hat{I},U)$, then we must have $I(f(\phi)) = f(I(\phi))$ for each $f \epsilon U'$. Therefore $\phi \epsilon D_o(\hat{I},U)$. Hence $D(\hat{I},U) \subset D_o(\hat{I},U)$ and μ is a zero-one measure by Theorem 3.

(2) Assume that μ is a zero-one measure. Now $U' = \{f_n | f_n(y) = y^n$ for each $y \epsilon U$, $n=0,\pm1,\pm2,\cdots\}$ ([21], p.366). Corollary 1 to Theorem 2 implies that there is an $x_\phi \epsilon X$ so that $I(f(\phi)) = f(\phi(x_\phi))$ for every $f \epsilon U'$, for each $\phi \epsilon D(\hat{I},U)$. But $I(f_1(\phi)) = I(\phi) = f_1(\phi(x_\phi)) = \phi(x_\phi)$. Therefore $\hat{I}(\phi) = I(\phi)$ on $D(\hat{I},U)$.

It remains to show that $D(\hat{I},U)$ is a subset of $D(I)$. If $\phi \epsilon D(\hat{I},U)$, then $f(\phi) \epsilon D(I)$ for every $f \epsilon U'$. Therefore $f_1(\phi) = \phi$ is in $D(I)$ and $D(\hat{I},U)$ is a subset of $D(I)$.

(c) (1) Assume that $\hat{I}$ is faithful. Recall that $U' = \hom(C_\bullet, C_\bullet) = \{i, 1\}$, where i is the identity map. If $\phi \epsilon D(I)$, then $I(f(\phi)) = f(I(\phi))$ for each $f \epsilon U'$. Therefore $\mu(X) = I(1(\phi)) = 1(I(\phi)) = 1$.

(2) Let $\mu(X) = 1$. It is easy to see that $D(\hat{I},U) = D(I)$. If $\phi \epsilon D(I)$, then $I[i(\phi)] = i[I(\phi)]$ and $I[\mathit{1}(\phi)] = \mathit{1}[I(\phi)]$. Hence $\hat{I}(\phi) = I(\phi)$ on $D(I)$.

(d) (1) Assume that $\hat{I}$ is faithful on $N(I)$. If ϕ is a characteristic function in (X,U), then ϕ belongs to $N(I)$. Therefore ϕ is in $D_o(\hat{I},U)$ and μ is a zero-one measure by Corollary 1 to Theorem 3.

(2) Assume that μ is a zero-one measure. If $\phi \epsilon N(I)$, then $\phi \epsilon D(\hat{I},U)$ because the members of U' are continuous and bounded, and $\mu(X) = 1$. Therefore $N(I)$ is a subset of $D(\hat{I},U) \cap D(I)$.

Let $\phi \epsilon N(I)$ be fixed. We will show that $I[f(\phi)] = f[I(\phi)]$ for every f in U'. Let ϕ_1 and ϕ_2 be the real and imaginary parts of ϕ, respectively. There are two sequences of simple functions S_n and T_n such that: (a) $S_n \to \phi_1$; (b) $T_n \to \phi_2$; (c) $|S_n| \leq |\phi_1| + 1$ for all n; and (d) $|T_n| \leq |\phi_2| + 1$ for all n. Note that $I(|\phi_i| + 1) < +\infty$ for $i=1,2$. Let $J_n = S_n + T_n i$ for $n=1,2,3,\cdots$. The dominated convergence theorem implies that $I(J_n) = I(S_n) + I(T_n)i \to I(\phi_1) + I(\phi_2)i = I(\phi)$.

Let $f \epsilon U'$ and let f_1, f_2 be the real and imaginary parts of f, respectively. For each n there is one and only one $A_n(B_n)$ in $\mu^{-1}(1)$ on which $S_n(T_n)$ is constant. Let a belong to the intersection of all the A_n and B_n. There is an N such that $J_n(a) \neq 0$ for $n \geq N$ because $J_n(a) \to \phi(a) \neq 0$. We will assume that $n \geq N$ for the rest of the proof.

The dominated convergence theorem implies that $f(\phi(a)) = f_1(\phi(a)) + f_2(\phi(a))i \leftarrow f_1(J_n(a)) + f_2(J_n(a))i = I(f_1(J_n)) + I(f_2(J_n))i \to I(f_1(\phi)) + I(f_2(\phi))i = I(f(\phi))$ because $|f_i(J_n)| \leq 1$ for all n and for $i=1,2$. But also $f(\phi(a)) \leftarrow f(J_n(a)) = f(I(J_n)) \to f(I(\phi))$ because $I(J_n) \to I(\phi)$. Hence $I(f(\phi)) = f(I(\phi))$ for each $f \epsilon U'$. $\square$

It is possible to derive convergence properties, absolute continuity, and other properties of $\hat{I}$ when μ is an arbitrary complex measure, but we will not do this here. We will complete this chapter by showing that $\hat{I}(\phi,\cdot)$ is not in general additive on A even if μ is a zero-one measure.

Let $\phi \epsilon D(\hat{I})$ and let A and B be disjoint measurable subsets of X such that $\hat{I}(\phi, A \cup B) = \hat{I}(\phi,A)\hat{I}(\phi,B)$. Then $I(f(\phi), A \cup B) = I(f(\phi),A) \cdot I(f(\phi),B)$ for every f in Y'. Therefore $\mu(A) + \mu(B) = \mu(A \cup B) = \mu(A)\mu(B)$ by choosing 1 for f. If $\mu(B)$ is not equal to one, then $\mu(A)$ must be equal to $\mu(B)/(\mu(B) - 1)$. Now assume that μ is a zero-one measure. If $\mu(A) = 1$, then $\mu(B) = 0$ and $\mu(A) + \mu(B)$ is not equal to $\mu(A)\mu(B)$. Therefore, if μ is a zero-one measure and $\mu(A) = 1$ $(\text{or } \mu(B) = 1)$, then $\hat{I}(\phi, A \cup B) \neq \hat{I}(\phi,A)\hat{I}(\phi,\beta)$ for every $\phi \epsilon D(\hat{I})$.

CHAPTER VI

EXAMPLES

This chapter consists of three examples. Each example is concerned with extending the Lebesgue integral to functions mapping a measure space into a space Y when the reals under addition (R), or the extended reals under addition ($\bar{R}$), is used as the range space of the functionals.

In Chapter IV we always assumed that the domain of the extension $D(\hat{I})$ was non-empty. In each example we have $D(\hat{I})$ non-empty and $D(\hat{I}) = D_o(\hat{I})$. Recall that $D_o(\hat{I})$ is the collection of all ϕ in $D(\hat{I})$ whose $\hat{I}(\phi)$ lies in Y. Examples 1 and 2 illustrate that it is possible to have many non-trivial members of CH(Y,R) or COH(Y,R), and to also have $D_o(\hat{I}) = D(\hat{I})$. The last example shows how $\hat{I}$ contains Aumann's integral [11] when Y consists of non-empty subsets of real numbers.

In all the examples, (X,A,μ) is a measure space with A a σ-algebra of subsets of X, and μ an extended real-valued complete measure on A. $I(\psi)$ denotes the Lebesgue integral of measurable real-valued, or extended real-valued, ψ over X with respect to μ. The domain of $I(\cdot)$ is $D(I) = \{\psi \mid |I(\psi)| < +\infty\}$.

Example 1

In this example Y will be a semi-group of operators under composition. Let E_n be n-dimensional Euclidean space and let Y = {n×n matrices $A|\det(A)$ is not zero$\}$. Y, with the relative product topology from (E_n,E_n), is a group and a topological space. The operation $\oplus$ on Y is matrix multiplication.

If $Y' = \{\log_k|\det(A)|\,\big|\,k>0 \text{ and } k\neq 1\}$, then Y' does not separate points in Y but Y' is a subset of CH(Y,R). This last assertion follows from $\det(A_1A_2) = \det(A_1)\det(A_2)$ and the fact that $\det(A)$ is a continuous function of its elements a_{ij}.

We will write $\phi(x) = A(x)\epsilon Y$ for a $\phi\epsilon(X,Y)$. Since $\log_k(a) = \ln(a)/\ln(k)$ for each $a>0$, we see that $D(\hat{I}) = \{\phi(x)=A(x)\epsilon(X,Y)\,\big|\,|I\left(\ln|\det(A(x))|\right)| < +\infty\}$. Hence $D(\hat{I})$ is not empty.

We now show that $D(\hat{I}) = D_o(\hat{I})$. If $\phi\epsilon D(\hat{I})$, then $\hat{I}(\phi) = \{A\epsilon Y\,\big|\,|\det(A)| = \exp(\alpha)\}$ where $\alpha = I(\ln|\det(A(x))|)$. This can be seen by considering the following equation: $I(\log_k|\det(A(x))|) = I(\ln|\det(A(x))|)/\ln k = \ln(\exp(\alpha))/\ln k = \log_k(\exp(\alpha))$.

Recall that $B = \cap\{f^{-1}(o)|f\epsilon Y'\}$. Obviously $B = \{A\epsilon Y|\det(A) = \pm 1\}$. If $\phi\epsilon D(\hat{I})$, then we know from Chapter IV that $\hat{I}(\phi) = A\oplus B$, where $\det(A) = \exp(\alpha)$, because Y is a group. Let $\gamma = \exp(\alpha/n)$ and $D = (d_{ij})$ where $d_{ii} = \gamma$ and $d_{ij} = 0$ if $i\neq j$. Then $\hat{I}(\phi) = D\oplus B = \{DA|\det(A) = \pm 1\} = \{\gamma A|\det(A) = \pm 1\}$.

Other properties of $\hat{I}$ can be found in Sections 1, 2, 4, and 5 of Chapter IV.

Example 2

Here Y will be a collection of functions but we will not use composition for $\oplus$. Let S be a closed interval of real numbers [a,b] and let Y = (S,R) with the product topology. Choose any $H\epsilon(R,R)$ which is one-to-one, onto, and continuous. For each g_1, g_2 in Y define $g_1 \oplus g_2 = H^{-1}\big(H(g_1) + H(g_2)\big)$. For example, if H is the identity function, then $\oplus$ is ordinary pointwise addition. For each $r \in S$ define $f_r\colon Y \to R$ as $f_r(g) = H\big(g(r)\big)$ and set $Y' = \{f_r \mid r \in S\}$. Then Y is a groupoid topological space, Y' separates points in Y, and Y' is a subset of CH(Y,R).

For each $\phi \in (X,Y)$ we will write $\phi(x) = g(x,t)$ where $g(x,t) \in (S,R)$ for each fixed x in X. We will use $d\mu(x)$ to denote integration with respect to x. $D(\hat{I})$ is not empty because $D(\hat{I}) = \{\phi \in (X,Y) \mid \left|I\left\{H\big(g(x,r)\big)d\mu(x)\right\}\right| < +\infty$ for each r in S}.

We now show that $D(\hat{I}) = D_o(\hat{I})$. Let $\phi \in D(\hat{I})$. The equation

$$I\big(f_r(\phi)\big) = I\left\{H\big(g(x,r)\big)d\mu(x)\right\} = H\left\{H^{-1}\left\{I\left\{H\big(g(x,r)\big)d\mu(x)\right\}\right\}\right\} =$$

$$f_r\left\{H^{-1}\left\{I\left\{H\big(g(x,t)\big)d\mu(x)\right\}\right\}\right\} \text{ implies that } \hat{I}(\phi) = H^{-1}\left\{I\left\{H\big(g(x,t)\big)d\mu(x)\right\}\right\}.$$

Now let H be the identity function. If $g_1, g_2 \in Y$, then define $g_1 \leq g_2$ if and only if $g_1(t) \leq g_2(t)$ for every t in S. Then $\hat{I}(\phi) = I\big(g(x,t)d\mu(x)\big)$, Y is a ℓ.o. group topological space, and $Y' \subset COH(Y,R)$. When H is the identity map other properties of $\hat{I}$ can be found in Sections 1 through 6 of Chapter IV.

Example 3

In this example we will discuss the relationship between $\hat{I}$ and Aumann's integral when Y consists of non-empty subsets of real numbers.

Let $P(R)$ be the set of all subsets of real numbers, Y is the set of all non-empty subsets of real numbers, and Y_c is the collection of all the non-empty convex subsets of real numbers. $\bar{R}$ denotes the extended real numbers under addition (with the usual conventions), $\oplus$ on Y and Y_c is defined as $A \oplus B = \{a+b \mid a \in A \text{ and } b \in B\}$, and $\leq$ is set inclusion. Y and Y_c are semi-groups under $\oplus$ and p.o. sets under $\leq$.

We will now define a topology on $P(R)$ and give Y and Y_c the relative topology. Let $2 = \{0,1\}$ with the discrete topology and give $(R,2)$ the product topology. Then $(R,2)$ is a compact Hausdorff topological space. Define $H:(R,2) \to P(R)$ as $H(f) = \{x \in R \mid f(x) = 1\}$. The natural topology N on $P(R)$ is $N = \{H(A) \mid A \text{ is open in } (R,2)\}$.

<u>Theorem 1</u>. Net A_β in $P(R)$ converges to A in $P(R)$ in the natural topology if and only if $\bigcap_\beta \bigcup_{\alpha \geq \beta} A_\alpha = \bigcup_\beta \bigcap_{\alpha \geq \beta} A_\alpha = A$.

<u>Proof</u>

Let us write $\underline{A}_\beta = \bigcup_\beta \bigcap_{\alpha \geq \beta} A_\alpha$, $\bar{A}_\beta = \bigcap_\beta \bigcup_{\alpha \geq \beta} A_\alpha$, and K_A for the characteristic function of A in $P(R)$. Since H is a homeomorphism of $(R,2)$ onto $P(R)$ we have $A_\beta \to A$ with respect to N if and only if $K_{A_\beta} \to K_A$ in $(R,2)$. But $K_{A_\beta} \to K_A$ if and only if $K_{A_\beta}(x) \to K_A(x)$ for every x in R. Now $K_{A_\beta}(x) \to K_A(x)$ for each x in R if and only if: (1) for each x in A there is a β_o such that $x \in A_\beta$ for $\beta \geq \beta_o$; and (2) for each x in A^c there is a β_1 so that $x \in A_\beta^c$ for $\beta \geq \beta_1$. Hence $A_\beta \to A$ with respect to N if and only if $\bar{A}_\beta \subset A$ and $A \subset \underline{A}_\beta$. But $\underline{A}_\beta \subset \bar{A}_\beta$ is always true. Therefore $A_\beta \to A$ with respect to N if and only if $\bar{A}_\beta = A = \underline{A}_\beta$. $\square$

For each A in Y define $f_u(A) = \sup(A)$ and $f_\ell(A) = \inf(A)$. In Appendix C we show that f_u and $-f_\ell$ belong to $OH(Y,\bar{R})$, and f_u and $-f_\ell$ belong to $COH(Y_c,\bar{R})$. The following example shows that f_u does not belong to $C(Y,\bar{R})$. If $A_n = \{-1\}\cup(1-1/n,1)$ for $n=1,2,3,\cdots$, then $A_n \to \{-1\}$ but $f_u(A_n)$ is one for all n.

If $X = [0,1]$, $\mathcal{A}$ is the Lebesgue measurable subsets of X, and if μ is Lebesgue measure, then we can extend the Lebesgue integral to functions mapping X into Y when $Y' = \{f_u,f_\ell\}$ and $\bar{R}$ is the range space of the functionals. We will show that this extension contains Aumann's integral, but first we will define Aumann's integral and discuss a few of its properties.

Aumann's Integral

Let $X = [0,1]$, let $\mathcal{A}$ be the Lebesgue measurable subsets of X, let μ be Lebesgue measure, and let E_n be n-dimensional Euclidean space.

Definition 1. If for each t in X, F(t) is a non-empty subset of E_n, then $A(F) = \{I(g)|g$ is an integrable function on X into E_n such that $g(t)\epsilon F(t)$ for every t in X$\}$.

A(F) is called Aumann's integral of F over X. It is possible that A(F) is empty. The following definition gives us the terminology needed to discuss the properties of A(F).

Definition 2. Assume that F maps X into non-empty subsets of E_n.

(a) If $x\epsilon E_n$, then $|x| = (|x_1|,\cdots,|x_n|)$. If $x,y\epsilon E_n$, then $x\le y$ if and only if $x_i \le y_i$ for $i=1,2,\cdots,n$.

(b) F is <u>Borel measurable</u> if and only if its graph $\{(t,x) | t \in X$ and $x \in F(t)\}$ is a Borel subset of $X \times E_n$.

(c) F is <u>integrably bounded</u> if and only if there is an integrable $h \in (X, E_n)$ such that $|x| \le h(t)$ for every $x \in F(t)$ and for almost all $t \in X$.

(d) $\tilde{F}(t)$ is the convex hull of $F(t)$ for each t in X.

Aumann [11] notes that $A(F)$ is always a convex subset of E_n. He also proves that if F is Borel measurable and integrably bounded then $A(F)$ is not empty and $A(F) = A(\tilde{F})$. Aumann shows that neither Borel measurability nor integrably bounded can be omitted in the previous statement. For the rest of this example assume that F is Borel measurable and integrably bounded.

We now show that if $n=1$, then $A(F) \in \hat{I}(F)$. The range space of F is Y, $Y' = \{f_u, f_\ell\}$, and the range space of the functionals is $\bar{R}$. In Appendix C we prove that the functions $F_\ell = f_\ell(F)$ and $F_u = f_u(F)$ are measurable and integrable on X into $\bar{R}$. Let $\alpha = I(F_\ell)$ and let $\beta = I(F_u)$. Then $\hat{I}(F) = \{A \in Y | f_\ell(A) = \alpha$ and $f_u(A) = \beta\}$. Also in Appendix C it is shown that $A(F) \in \{[\alpha,\beta],[\alpha,\beta),(\alpha,\beta],(\alpha,\beta)\} = Q$. Therefore $A(F) \in \hat{I}(F)$.

Now let us consider $\tilde{F}$ when $n=1$. The range space of $\tilde{F}$ is Y_C, $Y'_C = \{f_u, f_\ell\}$, and $\bar{R}$ is the range space of the functionals. It follows that $F_\ell = f_\ell(\tilde{F})$ and $F_u = f_u(\tilde{F})$. Therefore $A(F) = A(\tilde{F}) \in \hat{I}(\tilde{F}) = Q$.

Finally we note that $D_o(\hat{I}) = D(\hat{I})$ if Y or Y_C is the range space of the functions to be integrated and $Y' = Y'_C = \{f_u, f_\ell\}$. If $\phi \in D(\hat{I})$, then let $c = I(f_\ell(\phi))$ and $d = I(f_u(\phi))$. Then $\phi \in D_o(\hat{I})$ because $\hat{I}(\phi) = \{A \in Y(Y_C) | f_\ell(A) = c$ and $f_u(A) = d\}$.

APPENDIX

APPENDIX A

In this appendix we will give a proof of the following theorem, used in Chapter IV, because we were unable to find a proof of it in the literature.

Assume that $(X,\mathcal{A},\mu)$ is a measure space with $\mathcal{A}$ a σ-algebra of subsets of X and μ an extended real-valued complete measure on $\mathcal{A}$. $I(\psi,A)$ denotes the Lebesgue integral of measurable real-valued ψ over measurable A with respect to μ. We write $I(\psi)$ for $I(\psi,X)$. Let $f_n \in (X,R)$ and $|I(f_n)| < +\infty$ for $n=1,2,3,\cdots$, and let $f \in (X,R)$.

Theorem 1. If $\lim_n I(f_n,A)$ exists and is finite for every A in $\mathcal{A}$, and either

(a) f_n converges to f in measure, or

(b) $f_n \to f$ a.e. and μ is σ-finite,

then $|I(f)| < +\infty$ and $\lim_n I(f_n,A) = I(f,A)$ for every A in $\mathcal{A}$.

We will use the following two theorems in the proof of Theorem 1. Let $\pi_n(A) = I(f_n,A)$ and $\tau_n(A) = I(|f_n|,A)$ for each n and each measurable A. $|\pi_n|$ denotes the total variation of the finite signed measure π_n for $n=1,2,3,\cdots$.

Theorem 2. ([29], p.182). If $\lim_n I(f_n,A)$ exists and is finite for every measurable A, then:

(a) given any $\epsilon>0$, there is a $\delta>0$ so that $\mu(A) < \delta$ implies

$|\pi_n|(A) < \epsilon$ for all n;

(b) if the decreasing sequence of measurable sets A_m converges to the empty set and $\epsilon>0$ is given, then there is a positive integer M so that $|\pi_n|(A_m) < \epsilon$ for all $m \geq M$ and all n;

(c) there is a $g \epsilon (X,R)$ so that $|I(g)| < +\infty$ and $I(f_n,A) \to I(g,A)$ for every measurable A.

Theorem 3. ([28], p.108). We have $I(|f_n-f|) \to 0$ and $|I(f)| < +\infty$ if and only if:

(a) f_n converges to f in measure;

(b) given any $\epsilon>0$, there is a $\delta>0$ so that $\mu(A) < \delta$ implies $\tau_n(A) < \epsilon$ for all n;

(c) if the decreasing sequence of measurable sets A_m converges to the empty set and $\epsilon>0$ is given, then there is a positive integer M so that $\tau_n(A_m) < \epsilon$ for all $m \geq M$ and all n.

Proof of Theorem 1

(a) We first show that f_n convergent in measure to f implies the conclusion. Since $|\pi_n| = \tau_n$ for all n ([28], p.123), Theorems 2 and 3 imply that $I(|f_n-f|) \to 0$ and $|I(f)| < +\infty$.

Let $A \epsilon \mathcal{A}$ be fixed. For each n define g(n,A) to be f_n-f on A and zero on A^c. Then $|I(g(n,A))| < +\infty$ for all n, g(n,A) converges to zero in measure, and $\lim_n I(g(n,A),E)$ exists and is finite for every measurable E. Theorems 2 and 3 imply that $|I(f_n,A) - I(f,A)| \leq I(|g(n,A)|) = I(|g(n,A) - 0|) \to 0$. Hence $I(f_n,A) \to I(f,A)$.

(b) Now assume that $f_n \to f$ a.e. and $\mu(X) < +\infty$. It follows that f_n converges to f in measure. The conclusion follows from part (a) above.

(c) Assume that μ is σ-finite and that $f_n \to f$ a.e. Let $\{A_n \mid n=1,2,\cdots\}$ be a measurable partition of X so that each A_n has finite measure.

Theorem 2 says that there is a $g \in (X,R)$ such that $|I(g)| < +\infty$ and $I(f_n,A) \to I(g,A)$ for every measurable A. We will show that $g = f$ a.e.

Let A_n be fixed. Part (b) above implies that $I(f_n,E) \to I(f,E)$ for every measurable subset E of A_n. Therefore $I(f,E) = I(g,E)$ for every measurable subset E of A_n. Hence $f = g$ a.e. on A_n. It follows that $f = g$ a.e. on X. $\square$

APPENDIX B

In this appendix we collect together the basic facts about complex measures and integration with respect to complex measures that are needed in Chapter V. X is a non-empty set, A is a σ-algebra of subsets of X, and C is the set of complex numbers. $I(gd\tau,A)$ denotes the Lebesgue integral of measurable real-valued g over $A \in A$ with respect to real-valued measure τ.

Definition 1. A function μ on A into C is a complex measure if and only if: (1) $\mu(\emptyset) = 0$; and (2) if $E_i \in A$ for $i=1,2,\cdots$, and $E_i \cap E_j = \emptyset$ for $i \neq j$, then $\mu\left(\bigcup_{i=1}^{\infty} E_i\right) = \sum_{i=1}^{\infty} \mu(E_i)$.

Remarks on Definition 1

(1) Since $\bigcup_{i=1}^{\infty} E_i$ is not changed if the subscripts are permuted, every rearrangement of the series must also converge. Therefore the series converges absolutely.

(2) Royden ([18], pp.212-213), Rudin ([30], p.117), Hewitt and Ross ([21], p.118), and Zaanen ([29], p.178) define a complex measure in exactly the same way, but Halmos ([28], p.120) and Taylor ([22], p.375) define it in a slightly different manner. Halmos and Taylor define a complex measure as a function μ on A into the extended complex numbers so that $\mu = m_1 + m_2 i$ where m_1 and m_2 are signed measures (Definition 2).

(3) Note that $|\mu(X)| < +\infty$. Rudin ([30], p.118) shows that the range of μ lies in a disk of finite radius.

Definition 2

(1) A function m on $\mathcal{A}$ into the extended real numbers is a signed measure if and only if: (a) $m(\emptyset) = 0$; (b) m assumes at most one of the values $+\infty$ and $-\infty$; and (c) if $E_i \in \mathcal{A}$ for $i=1,2,\cdots$, and $E_i \cap E_j = \emptyset$ for $i \neq j$, then $m(\bigcup_{i=1}^{\infty} E_i) = \sum_{i=1}^{\infty} m(E_i)$.

(2) If m is a signed measure on $\mathcal{A}$, then $m^+(A) = \sup\{m(E) \mid E \subset A, E \in \mathcal{A}\}$ and $m^-(A) = -\inf\{m(E) \mid E \subset A, E \in \mathcal{A}\}$ for each $A \in \mathcal{A}$.

Remarks on Definition 2. If m is a signed measure on $\mathcal{A}$, then m^+ and m^- are measures on $\mathcal{A}$ with $m = m^+ - m^-$. If μ is a complex measure on $\mathcal{A}$, then there are finite signed measures m_1 and m_2 so that $\mu = m_1 + m_2 i$. Define $\mu_1 = m_1^+$, $\mu_2 = m_1^-$, $\mu_3 = m_2^+$, and $\mu_4 = m_2^-$. Then the μ_j are finite measures and $\mu = (\mu_1 - \mu_2) + (\mu_3 - \mu_4)i$.

Definition 3. If $f = f_1 + f_2 i$ maps X into C, then f is measurable if and only if both f_1 and f_2 are measurable real-valued functions.

Definition 4. Let $\mu = (\mu_1 - \mu_2) + (\mu_3 - \mu_4)i$ be a complex measure and let $A \in \mathcal{A}$.

(1) $D(I,A) = \{f \in (A,C) \mid f = f_1 + f_2 i$ is measurable on A and $|I(f_i d\mu_j, A)| < +\infty$ for $i=1,2$ and $j=1,2,3,4\}$. $D(I) = D(I,X)$.

(2) If $f \in D(I,A)$, then let $ij = I(f_i d\mu_j, A)$ for $i=1,2$ and $j=1,2,3,4$. Then

$$I(f,A) = (11-12-23+24) + (13-14+21-22)i. \tag{1}$$

We write $I(f)$ for $I(f,X)$.

<u>Remarks on Definition 4</u>

(1) $I(f,A)$ is the Lebesgue integral of measurable complex-valued f over A with respect to complex measure μ. The definition of $I(f,A)$ comes from Zaanen ([29], pp. 178-179) and Taylor ([22], p.375), and it is also contained in Halmos ([28], p.124) and Loève ([23], p. 142).

(2) If μ is real-valued ($\mu=\mu_1$), then $I(f,A) = I(f_1 d\mu,A) + I(f_2 d\mu,A)i$.

(3) Let $\mu = m_1 + m_2 i = (\mu_1-\mu_2) + (\mu_3-\mu_4)i$. If we define $I(fdm_1,A) = I(fd\mu_1,A) - I(fd\mu_2,A)$ and $I(fdm_2,A) = I(fd\mu_3,A) - I(fd\mu_4,A)$ for $f\epsilon D(I,A)$, then $I(f,A) = I(fdm_1,A) + I(fdm_2,A)i$.

<u>Properties of I(f,A)</u>

We list below the basic properties of $I(f,A)$ that are used in Chapter V.

(1) If $A_1,A_2\epsilon\mathcal{A}$, $A_1\subset A_2$, and if $f\epsilon D(I,A_2)$, then $f\epsilon D(I,A_1)$.

(2) If $f\epsilon D(I,A)$, then $|I(f,A)| < +\infty$.

(3) If $f_1,f_2\epsilon D(I,A)$ and if $z_1,z_2\epsilon C$, then $z_1f_1 + z_2f_2\epsilon D(I,A)$ and $I(z_1f_1+z_2f_2,A) = z_1I(f_1,A) + z_2I(f_2,A)$.

(4) If $A,B\epsilon\mathcal{A}$, $A\cap B = \emptyset$, and if $f\epsilon D(I,A\cup B)$, then $I(f,A) + I(f,B) = I(f,A\cup B)$.

(5) If $A\epsilon\mathcal{A}$, $f\epsilon(A,C)$, f is measurable on A, and if $|f(x)| \le M$, some $M>0$, on A, then $f\epsilon D(I,A)$.

(6) Let $A_i\epsilon\mathcal{A}$ for $i=1,2,\cdots,n$, and $A_i\cap A_j = \emptyset$ for $i\neq j$. If $\phi(x) = z_i\epsilon C$ on A_i for $i=1,2,\cdots,n$, and if $A = \bigcup_{i=1}^{n} A_i$, then $\phi\epsilon D(I,A)$ and $I(\phi,A) = \sum_{i=1}^{n} z_i\mu(A_i)$.

<u>Remark</u>. Properties (2), (5), and (6) above are very important in the development of Chapter V. Property (2) states that $I(\cdot,A)$ maps $D(I,A)$ into C, (5) says that all bounded measurable functions belong to $D(I,A)$, and (6) indicates that all simple functions belong to $D(I,A)$. All of these properties follow directly from the definition of $I(f,A)$ and the fact that all the measures μ_j are <u>finite</u>. This is the reason why we have chosen our definition of a complex measure and not the one given by Halmos.

APPENDIX C

There are four theorems in this appendix. Each theorem is used only in Example 3 in Chapter VI.

R denotes the real numbers and $\bar{R}$ the extended real numbers under addition. Let Y be all the non-empty subsets of R and let Y_c be all the convex members of Y. Addition in Y is $A \oplus B = \{a+b \mid a \in A, b \in B\}$, $-A = \{-a \mid a \in A\}$, and $\leq$ is set inclusion. Define $f_u(A) = \sup(A)$ and $f_\ell(A) = \inf(A)$ for each A in Y. If A_β is a net in Y, then $\underline{A}_\beta = \bigcup_\beta \bigcap_{\alpha \geq \beta} A_\alpha$ and $\bar{A}_\beta = \bigcap_\beta \bigcup_{\alpha \geq \beta} A_\alpha$. A net A_β in Y converges to A in Y if and only if $\underline{A}_\beta = A = \bar{A}_\beta$ (Theorem 1 in Chapter VI).

<u>Theorem 1</u>. f_u and $-f_\ell$ belong to $OH(Y,\bar{R})$.

<u>Proof</u>. Obviously $f_u, -f_\ell \in O(Y,\bar{R})$. Since $-(A \oplus B) = (-A) \oplus (-B)$ and $f_\ell(A) = -f_u(-A)$, it suffices to show that $f_u \in H(Y,\bar{R})$.

If $A, B \in Y$ and both are bounded above, then it is well known ([31], p.8) that $f_u(A \oplus B) = f_u(A) + f_u(B)$. If $A, B \in Y$ and A is unbounded above, then $A \oplus B$ is unbounded above and $f_u(A \oplus B) = +\infty = f_u(A) = f_u(A) + f_u(B)$. Note that $f_\ell(\cdot) < +\infty$ and $f_u(\cdot) > -\infty$ because their domain consists of non-empty subsets of R. Therefore $(+\infty) + (-\infty)$ can never occur. □

<u>Theorem 2</u>. f_u and $-f_\ell$ belong to $COH(Y_c,\bar{R})$.

<u>Proof</u>. Theorem 1 implies that f_u and $-f_\ell$ belong to $OH(Y_c,\bar{R})$. Let net A_β in Y_c converge to A in Y_c. Since $-A_\beta \to -A$ and $f_\ell(A_\beta) = -f_u(-A_\beta)$ it

suffices to show that $f_u(A_\beta) \to f_u(A)$. Let $x_\beta = f_u(A_\beta)$ and let $x = f_u(A)$.

(a) Assume that $x < +\infty$ and let $\delta > 0$ be given. We will show that there is a β_o so that $x-\delta \le x_\beta \le x+\delta$ for $\beta \ge \beta_o$.

Let $t \in A$ so that $x-\delta < t \le x$. Then there is a β_1 so that $t \in A_\beta$ for $\beta \ge \beta_1$. Therefore $x-\delta < t \le x_\beta$ for $\beta \ge \beta_1$.

There is a β_2 so that $x+\delta$ is not in A_β for $\beta \ge \beta_2$ because $x+\delta$ is not in A. Therefore $x+\delta >$ each a in A_β or $x+\delta <$ each a in A_β, for $\beta \ge \beta_2$, because the A_β are convex. We now show that there is a $\beta_3 \ge \beta_2$ so that $x+\delta >$ each a in A_β for $\beta \ge \beta_3$.

Assume that there is no $\beta_3 \ge \beta_2$ so that $x+\delta >$ each a in A_β for $\beta \ge \beta_3$. Let D be the directed set of the net. Then there is a cofinal $E \subset D$ so that $x+\delta <$ each a in A_β for all β in E. The net $(A_\beta | \beta \in E)$ is a subnet of the net A_β and therefore has the same limit. Therefore $x+\delta \le$ each a in $\lim_{\beta \in E} A_\beta = A$. This is a contradiction because $x+\delta > x = \sup(A)$.

Hence $x+\delta \ge x_\beta$ for $\beta \ge \beta_3$. Choose $\beta_0 \ge \beta_1$ and $\beta_0 \ge \beta_3$.

(b) Assume that $x = +\infty$ and let $M>0$ be given. We will show that there is a β_0 so that $x_\beta \ge M$ for $\beta \ge \beta_0$.

Let $t \in A$ so that $M \le t$. Then there is a β_0 so that $t \in A_\beta$ for $\beta \ge \beta_0$. Hence $M \le t \le x_\beta$ for $\beta \ge \beta_0$. □

For the next two theorems we introduce the following notation. Let $X = [0,1]$, $\mathcal{A}$ = the Lebesgue measurable subsets of X, $\mathcal{B}$ = the Borel subsets of X, μ = Lebesgue measure, $I(\psi, E)$ = the Lebesgue integral of measurable real-valued ψ over E in $\mathcal{A}$ with respect to μ, and let

$I(\psi) = I(\psi,X)$. An F in (X,Y) is <u>Borel measurable</u> if and only if its graph $G(F) = \{(x,y)|x\epsilon X, y\epsilon F(x)\}$ is a Borel subset of $X\times R$. An F in (X,Y) is <u>integrably bounded</u> if and only if there is an h in (X,R), $|I(h)| < +\infty$, so that $|x| \le h(t)$ for every $x\epsilon F(t)$ and for almost all $t\epsilon X$. For the rest of this appendix we will assume that F is Borel measurable and integrably bounded. Define $F_u = f_u(F)$ and $F_\ell = f_\ell(F)$. Aumann's integral of F over X is $A(F) = \{I(g)|g$ is an integrable function on X into R such that $g(t)\epsilon F(t)$ for every t in X$\}$.

<u>Theorem 3</u>. F_u and F_ℓ are Lebesgue measurable functions on X into $\bar{R}$ and $|I(F_u)| < +\infty$, $|I(F_\ell)| < +\infty$.

<u>Proof</u>. If F_u and F_ℓ are Lebesgue measurable then $|I(F_u)| < +\infty$ and $|I(F_\ell)| < +\infty$ because F is integrably bounded. We will first show that F_u is Lebesgue measurable.

(1) (a) There is an $E\epsilon\mathcal{A}$ so that $F_u(x) \le h(x)$ for each $x\epsilon E^c$ and $\mu(E) = 0$. We may assume that $E\epsilon\mathcal{B}$ because there is an $E_1\epsilon\mathcal{B}$ so that $E\subset E_1$ and $\mu(E_1) = 0$ ([28], p.66). Define $F' = F$ on E^c, $F' = \{0\}$ on E, and $F'_u = f_u(F')$. Then F' is Borel measurable and integrably bounded, $F'_u(x) \le h(x)$ for all x in X, and $F'_u = F_u$ a.e. Hence F_u is Lebesgue measurable if F'_u is Lebesgue measurable. Therefore it suffices to consider only finite-valued F_u with $F_u(x) \le h(x)$ on X.

(b) Let $\beta = \sup A(F)$. Then $|\beta| < +\infty$ because A(F) is non-empty ([11], p. 2) and F is integrably bounded. There is a sequence g_i in (X,R) so that: (i) $|I(g_i)| < +\infty$ for all i; (ii) $I(g_i) \to \beta$; and (ii) $g_i(x) \epsilon F(x)$ for all i and all x in X. We now show that we can choose the g_i so that $g_1 \leq g_2 \leq \cdots \leq g_n \leq \cdots$.

Define $f_1 = g_1$ and $f_{n+1} = \sup\{f_1, \cdots, f_n, g_{n+1}\}$ for $n \geq 1$. Then: (i) $f_1 \leq f_2 \leq \cdots \leq f_n \leq \cdots$; (ii) $|I(f_i)| < +\infty$ all i; and (iii) $f_i(x) \epsilon F(x)$ for all i and all x in X. Since $g_i \leq f_i$ for all i, we have $I(g_i) \leq I(f_i) \leq \beta$ for all i. Hence $I(f_i) \to \beta$. Therefore we will assume that $g_1 \leq g_2 \leq \cdots \leq g_n \leq \cdots$.

(c) Now we will show that we may assume that the g_i are Borel measurable. For each g_i there is a Borel measurable f_i and an $E_i \epsilon \mathcal{B}$ so that $g_i = f_i$ on E_i^c and $\mu(E_i) = 0$ ([28], p.90). Let $E = \bigcup_{i=1}^{\infty} E_i$. Then $g_i = f_i$ for all i on E^c and $E \epsilon \mathcal{B}$ with $\mu(E) = 0$. Define $F' = F$ on E^c, $F' = \{0\}$ on E, and $F'_u = f_u(F')$. Also define $h_i = f_i$ on E^c and $h_i = 0$ on E for each i. Then each h_i is Borel measurable, F' is Borel measurable and integrably bounded, $\beta = \sup A(F')$, $|I(h_i)| < +\infty$ for all i, $I(h_i) \to \beta$, $h_i(x) \epsilon F'(x)$ for all i and all x in X, and $h_1 \leq h_2 \leq \cdots \leq h_n \leq \cdots$. Also $F'_u = F_u$ a.e. Hence, if F'_u is Lebesgue measurable, then so is F_u. Therefore we will assume that each g_i is Borel measurable.

(d) Let $g = \lim g_i$. Then $g(x) \leq F_u(x)$ for all x in X and g is Borel measurable. We claim that $g = F_u$ a.e. which implies that F_u is Lebesgue measurable. To show this let us suppose that there is an $E \in \mathcal{A}$, $\mu(E) > 0$, so that $g(x) < F_u(x)$ on E. We will use this to construct an $f \in (X,R)$ with the following properties: (i) $|I(f)| < +\infty$; (ii) $f(x) \in F(x)$ for all x in X; and (iii) $I(g) < I(f)$. This will give us a contradiction because, by the dominated convergence theorem ($|g_i| \le h$ for all i), $\beta = \lim I(g_i) = I(g) < I(f) \le \beta$.

(e) We may suppose that $E \in \mathcal{B}$ because there is an $E_1 \in \mathcal{B}$ so that $E_1 \subset E$ and $\mu(E) = \mu(E_1)$ ([28], p.66). Define $g^+ = \sup\{g,0\}$, $g^- = \sup\{-g,0\}$, $V(g^+) = \{(x,y) | x \in X \text{ and } 0 \le y \le g^+(x)\}$, and $W(g^-) = \{(x,y) | x \in X \text{ and } 0 \le y < g^-(x)\}$. Now $V(g^+)$ and $W(g^-)$ are Borel subsets of $X \times R$ ([28], pp. 142-143).

For any $A \subset X \times R$ we write $-A = \{(x,-y) | (x,y) \in A\}$. If A is a Borel subset of $X \times R$, then so is $-A$.

Let $T = \{(x,y) | x \in E \text{ and } y \ge 0\}$ and let $K = \left(T \cap \left(V(g^+)\right)^c\right) \cup \left(-\left(T \cap W(g^-)\right)\right)$. Then $\{(x,y) | x \in E \text{ and } y > g(x)\} = K$ and therefore is a Borel subset of $X \times R$.

Define $F' \in (X,Y)$ so that its graph $G(F') = (E^c \times \{0\}) \cup \left(K \cap G(F)\right)$. Then F' is Borel measurable and integrably bounded. We see that the values of F' are in Y because $g(x) < F_u(x)$ on E and $F_u(x) = \sup F(x)$ on X.

Therefore, ([11], p.3) there is a $g' \in (X,R)$ so that $|I(g')| < +\infty$ and $g'(x) \in F'(x)$ on X. Notice that $g' = 0$ on E^c and $g' > g$ on E.

Let $I(g',E) - I(g,E) = \delta > 0$ and let $\alpha = \delta/2$. Choose k so that $I(g,E^c) - \alpha < I(g_k,E^c) \leq I(g,E^c)$. Define $f = g_k$ on E^c and $f = g'$ on E. Then: (i) $I(g) < I(f)$; (ii) $f(x) \epsilon F(x)$ on X; and (iii) $|I(f)| < +\infty$.

(2) We now show that F_ℓ is Lebesgue measurable. Let $H \epsilon (X,Y)$ so that its graph is $-G(F)$. Then H is Borel measurable and integrably bounded. Let $H_u = f_u(H)$. Part (1) above implies that H_u is Lebesgue measurable. But $F_\ell = -H_u$. □

Let $\alpha = I(F_\ell)$ and $\beta = I(F_u)$.

<u>Theorem 4</u>. $A(F)$ is one of the intervals $[\alpha,\beta]$, or $(\alpha,\beta]$, or $[\alpha,\beta)$, or (α,β).

<u>Proof</u>. Theorem 3 implies that $I(F_u) = \sup A(F)$. Since $F_\ell = -f_u(-F)$ we see that $I(F_\ell) = -\sup[-A(F)] = \inf A(F)$. The result follows because $A(F)$ is a convex subset of R ([11], p.2). □

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